Exploring Structure Constants through the Hexagon Approach

Fishnet QFTs: Integrability, Periods and Beyond, University of Southampton

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Based on works with Arthur Klemenchuk Sueiro and Alessandro Georgoudis

Plan

Structure constants from hexagons

Interlude on 4d fishnet CFT

Application to small spin and strong coupling limits

Outlook

Hexagons

Hexagons

Idea: break n-point functions of local operators in planar limit in simpler pieces (hexagons)

[BB,Komatsu,Vieira]
[Fleury,Komatsu]
[Eden,Sfondrini]

$$F_n \quad \to \quad \mathcal{H}_1 \otimes \mathcal{H}_2 \ldots \otimes \mathcal{H}_{2(n-2)}$$

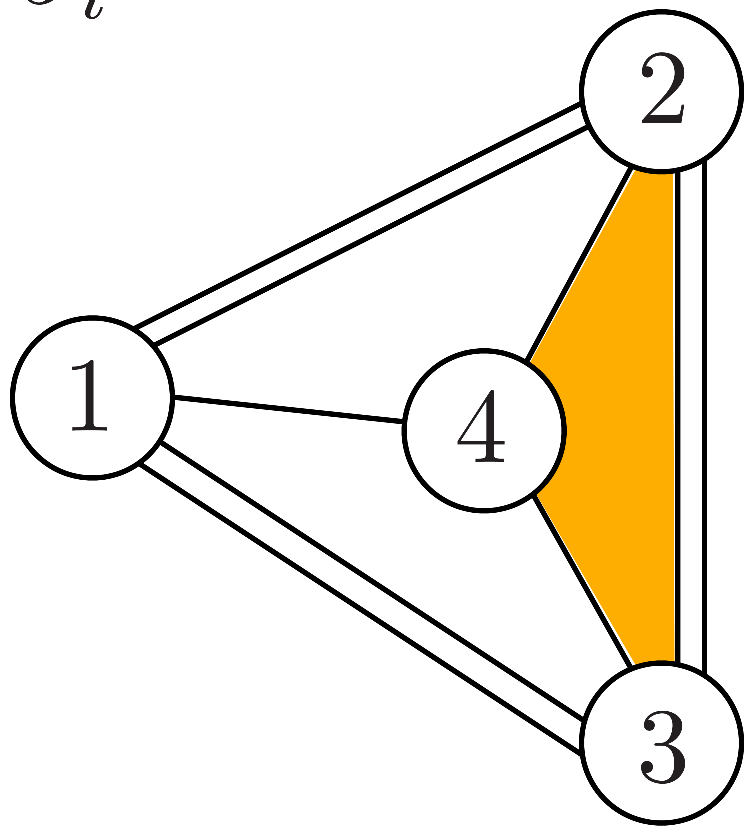
Decomposition is related to triangulation of punctured spheres

Ex. 4-point function of half-BPS operators

See also Grisha's talk

Each operator $\mathrm{Tr}\, Z_i^{J_i}$ can be viewed as a spin chain of length J_i

The planar correlator is triangulated into four hexagons
by following the pattern of Wick contractions



Hexagons

Idea: break n-point functions of local operators in planar limit in simpler pieces (hexagons)

[BB,Komatsu,Vieira]
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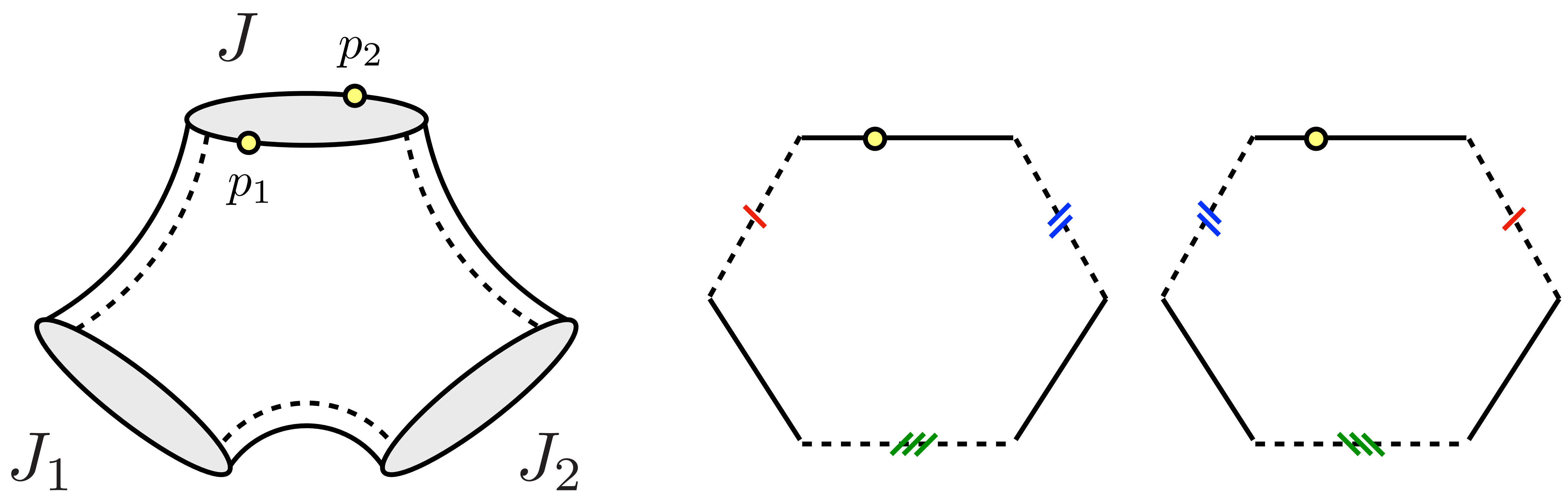
$$F_n \quad \rightarrow \quad \mathcal{H}_1 \otimes \mathcal{H}_2 \ldots \otimes \mathcal{H}_{2(n-2)}$$

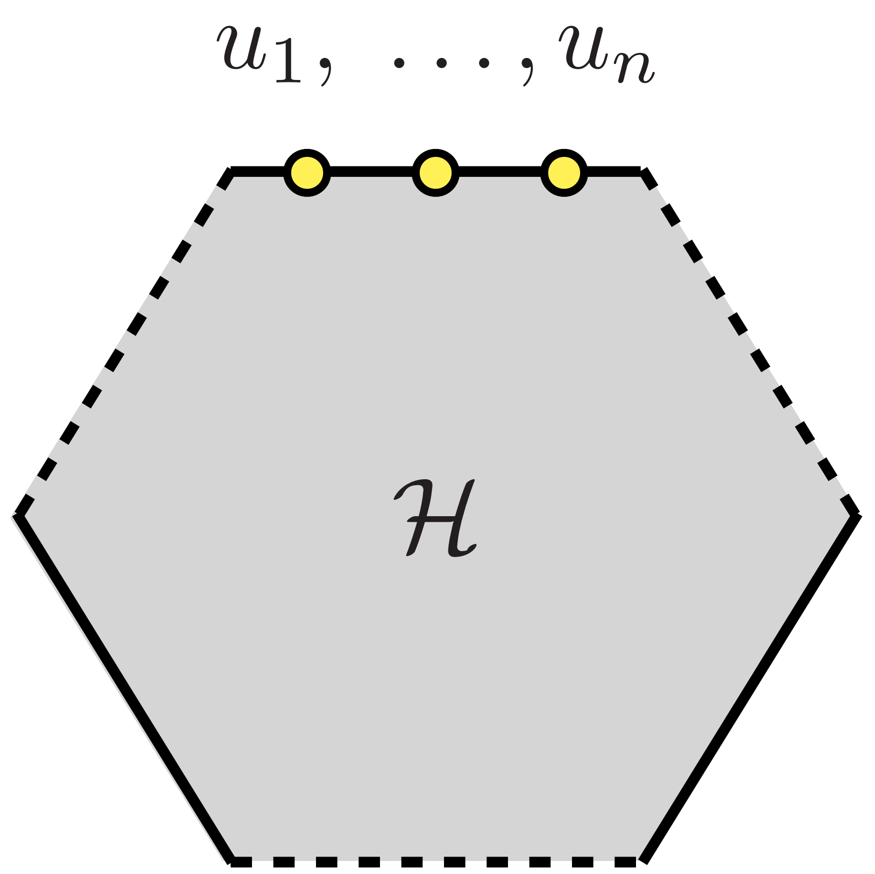
Decomposition is related to triangulation of punctured spheres

Ex. 3-point functions (structure constants)



Two hexagons suffice to cover both the front and back of the pair-of-pants geometry

Hexagons: in some detail



Hexagons are bounded by 3 spin chains and 3 mirror cuts

They describe form factors for absorption of magnons on edges

$$h(u_1, \ldots, u_n) = \langle \mathcal{H} | \chi(u_1) \ldots \chi(u_n) \rangle$$

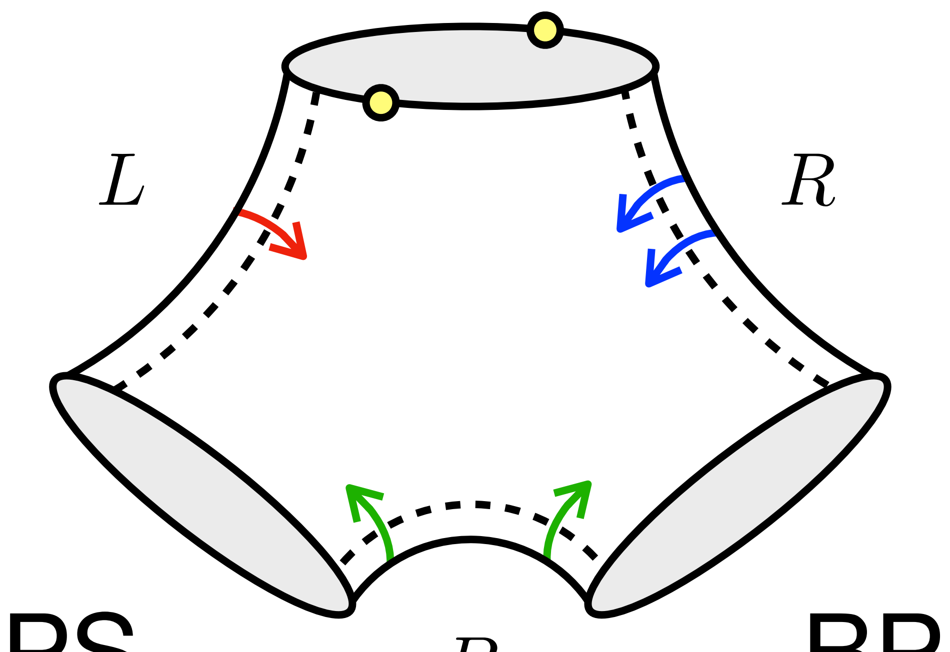
They obey stringent integrable bootstrap constraints that determine them at any coupling

Magnons on spin chains describe operators - those on mirror cuts describe fluctuations of the open strings stretching between two operators

Gluing procedure entails summing over 1) all possible ways of distributing physical magnons on the two hexagons and 2) a complete basis of mirror magnons along each mirror cut

Hexagons sums

Sum over a complete basis of mirror states along each seam



$$C^{\circ\circ\bullet} = \mathcal{N} \times \sum_L \sum_R \sum_B e^{-\ell_L \mathcal{E}_L - \ell_R \mathcal{E}_R - \ell_B \mathcal{E}_B} |\mathcal{H}|^2$$

Bridge lengths (count Wick contractions on each side)

$$\ell_L = \frac{J_1 + J - J_2}{2} \qquad \ell_B = \frac{J_1 + J_2 - J}{2} \qquad \ell_R = J - \ell_L$$

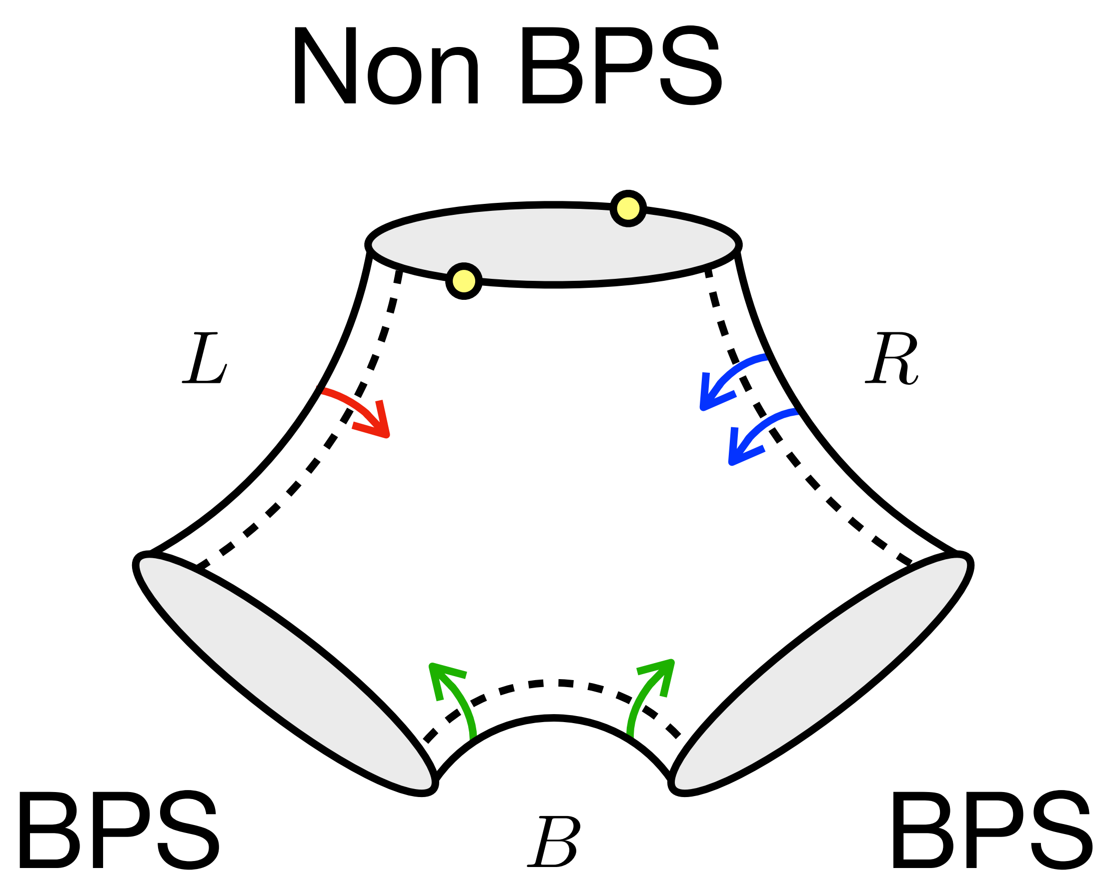
States are labelled by rapidities and bound states labels

$$\sum = \sum_{N=0}^{\infty} \prod_{i=1}^{N} \sum_{a_i=1}^{\infty} \int \frac{du_i}{2\pi} \mu_{a_i}(u_i) \prod_{i<j} p_{a_i,a_j}(u_i, u_j)$$

They determine $\mathfrak{su}(2|2)^2$ rep. and energy of the magnon $\mathcal{E}_a(u) = \log\left(x^{[+a]} x^{[-a]}\right)$

Hexagons sums

Sum over a complete basis of mirror states along each seam



$$C^{\circ\circ\bullet} = \mathcal{N} \times \sum_{L}\sum_{R}\sum_{B} e^{-\ell_L \mathcal{E}_L - \ell_R \mathcal{E}_R - \ell_B \mathcal{E}_B} |\mathcal{H}|^2$$

Integrand follows from hexagon form factors after summing over all flavours

[BB,Gonçalves,Komatsu,Vieira]

$$|\mathcal{H}|^2 = \prod_{i,j,k}^{N_L,N_R,N_B} \frac{\mathbb{W}^L_{a_i}(u_i)\mathbb{W}^R_{b_j}(v_j)\mathbb{W}^B_{c_k}(w_k)}{p_{a_i b_j}(u_i, v_j)}$$

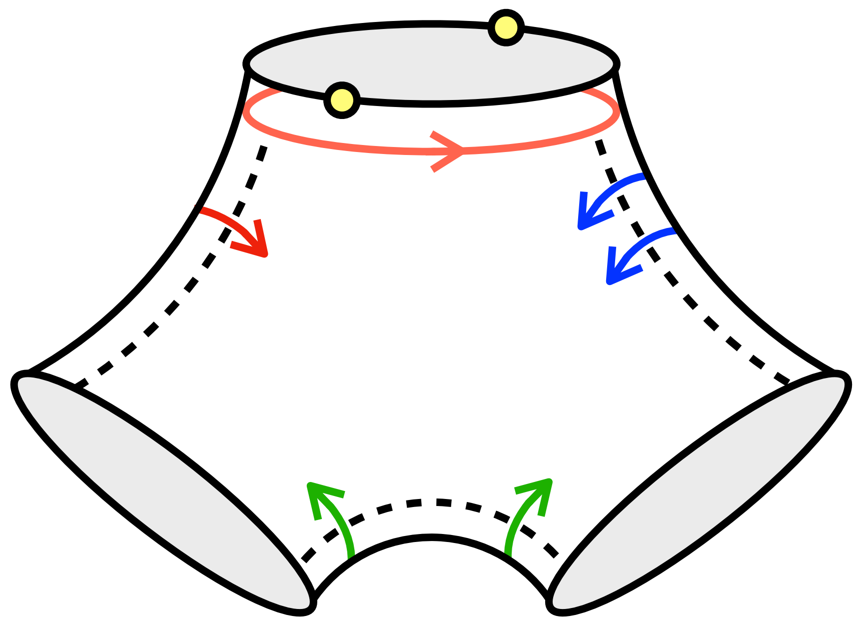
Remarks: Nicely it factorizes into adjacents (L&R) and bottom channel (B)

$$C^{\circ\circ\bullet} = \mathcal{N}(J) \times \mathcal{A}(\ell_L, \ell_R) \times \mathcal{B}(\ell_B)$$

The integrand develops a **double pole** when two magnons in adjacent channels share the same quantum numbers (same rapidity and bound state label)

$$p_{ab}(u,v) \propto (u-v)^2 \delta_{ab}$$

Hexagons sums



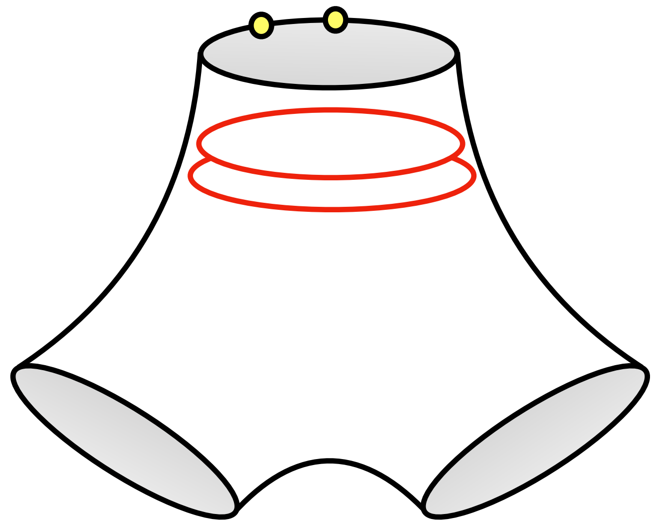
The method is valid up to wrapping order

Div. indicate that magnons wrapping around excited operator need a separate treament; similarly to UV divergences, a renormalization is needed

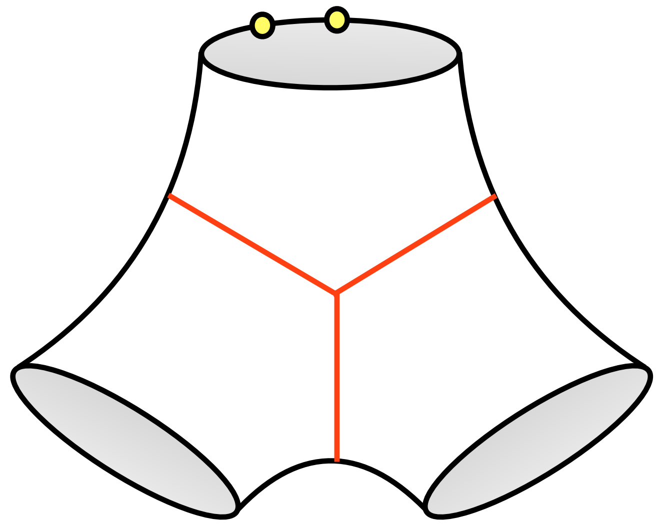
They produce terms ~ $e^{-J\mathcal{E}_a}$ reminiscent to wrapping effects from the TBA for the spectrum problem

Variety of “wrappings”

See also Didina’s talk



usual type



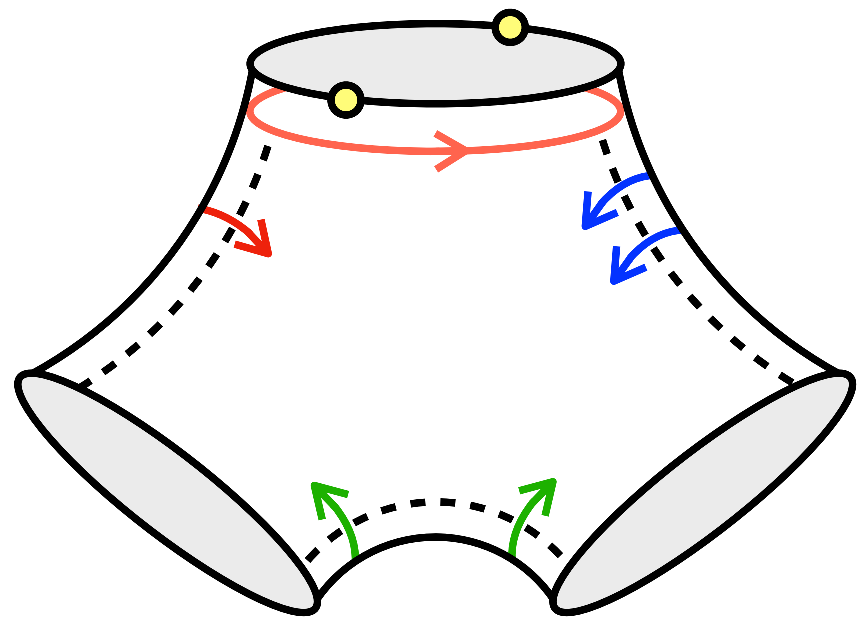
cubic type

General solution (for *all* wrappings) is still unknown for general structure constants

For a **single** non-BPS operator some simplifications emerge from Lüscher formula

[BB,Gonçalves,Komatsu]
[BB,Caetano,Fleury]

Conjecture for wrapping corrections



Overall structure conjectured to stay the same

$$C^{\circ\circ\bullet} = \mathcal{N} \times \sum_L \sum_R \sum_B e^{-\ell_L \mathcal{E}_L - \ell_R \mathcal{E}_R - \ell_B \mathcal{E}_B} |\mathcal{H}|^2$$

Weights of mirror magnons on mirror cuts absorb all the corrections (up to normalization factor)

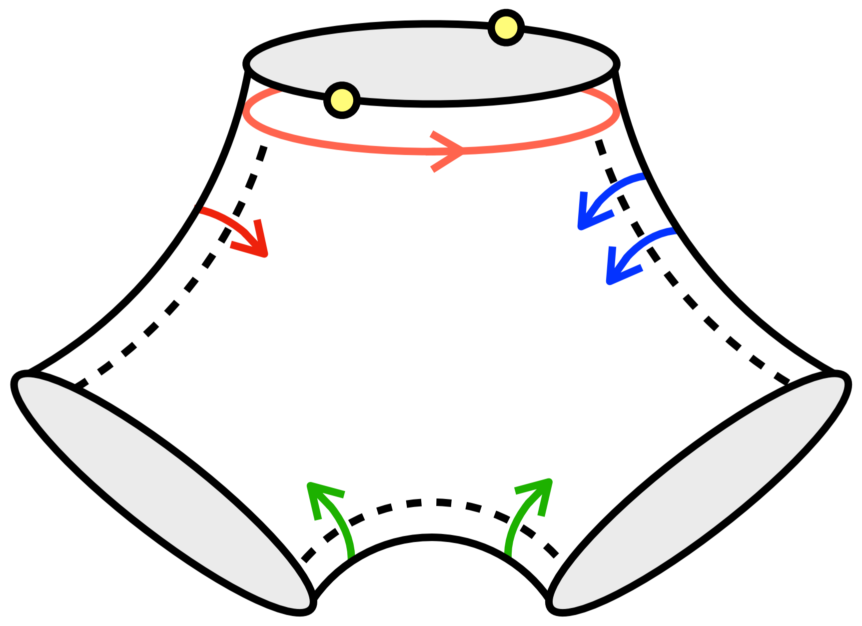
Asymptotic solution

Weights are given by transfer matrices of **asymptotic** (large charge) symmetry algebra

$\mathtt{su}(2|2)$ transfer matrices $\quad T_a(u) = \mathrm{str}\, S_{a1}(u, \mathbf{z})$ [BB,Gonçalves,Komatsu,Vieira]

Ex. $\quad \mathbb{W}_a^R(u) = \dfrac{T_a(u)}{h_{1a}(\mathbf{z}, u)} \qquad \mathbb{W}_a^L(u) = h_{a1}(u, \mathbf{z})\, T_a(u)$

Conjecture for wrapping corrections



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Asymptotic solution

Weights are given by transfer matrices of **asymptotic** (large charge) symmetry algebra

Full solution $\mathbb{W} = \mathbb{W}^{\text{asy}} + \delta\mathbb{W}$ (wrapping induced)

Weights are given by transfer matrices of **full** (finite charge) symmetry algebra

Examples in adjacent channels

Emerging relations

[BB,Caetano,Fleury]
[BB,Georgoudis,Klemenchuk]

$$e^{-L\mathcal{E}_a(u)}\mathbb{W}_a^L(u)\mathbb{W}_a^R(u) = \frac{Y_{a,0}}{1+Y_{a,0}}$$

$$\frac{\mathbb{W}_a^L(u)}{\mathbb{W}_a^R(u)} = p_{a1}(u,\mathbf{z})\, e^{i\sum_b \fint \frac{dv}{2\pi} L_b(v)\partial_v \log p_{ba}(v,u)}$$

with $Y_{a,0}$ the Y functions solving the TBA equations and $L_a = \log(1+Y_{a,0})$

Take a nicer form in terms of the super-conformal transfer matrices

[Gromov,Kazakov,Vieira]

$$Y_{a,s} = \mathbf{T}_{a,s+1}\mathbf{T}_{a,s-1}/\mathbf{T}_{a+1,s}\mathbf{T}_{a-1,s}$$

giving

$$\mathbb{W}_a^L(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)}\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}_{a,0}^-(u)} \qquad \mathbb{W}_a^R(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)}\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}_{a,0}^+(u)}$$

with

$$\mathbf{T}_{a,s}^{\pm} = \mathbf{T}_{a,s}(u \pm i/2)$$

[Gromov,Kazakov,Leurent,Volin]

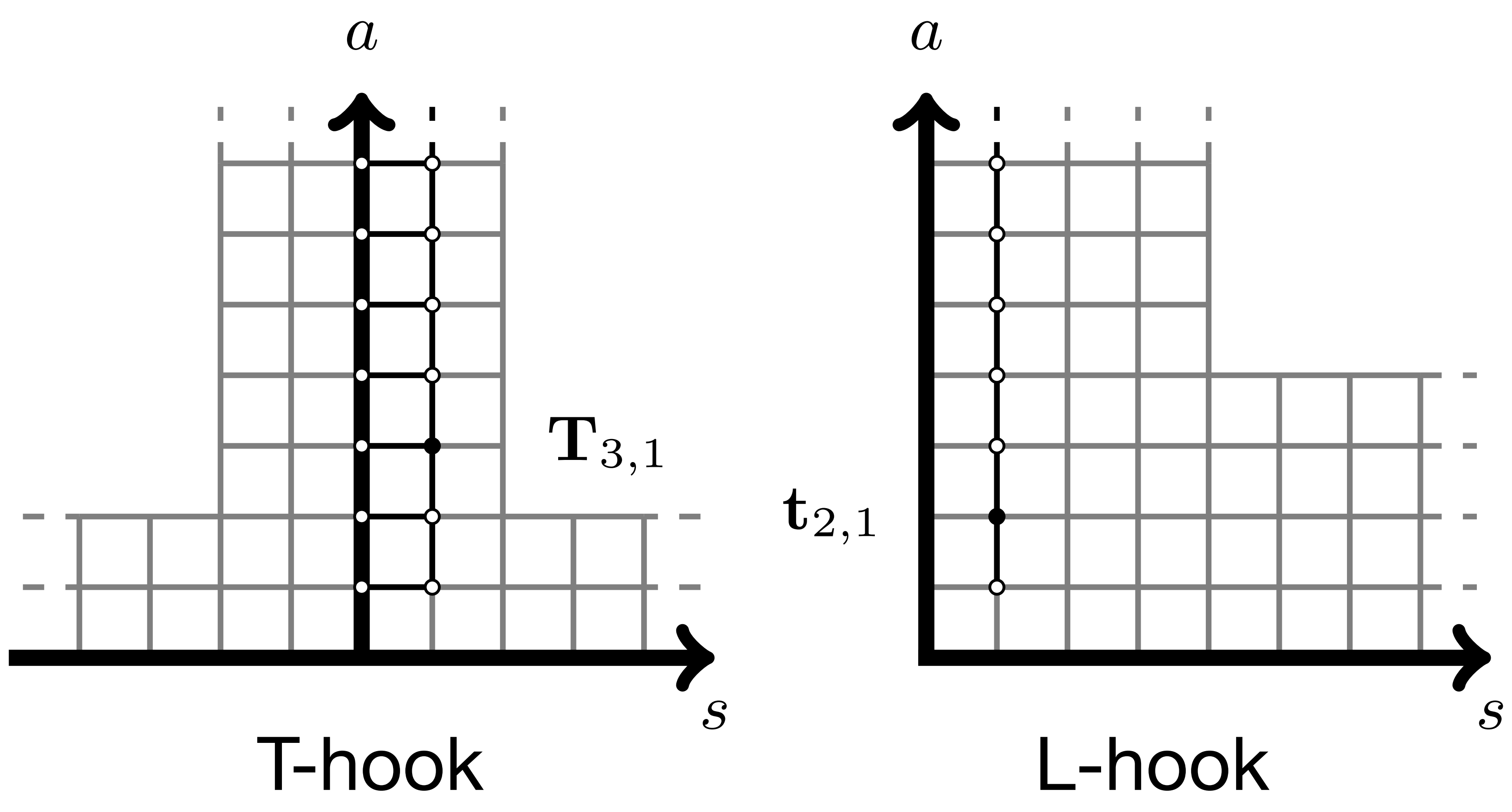
Conjecture

Exact weights

$$\mathbb{W}_a^L(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)}\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}_{a,0}^-(u)}$$

$$\mathbb{W}_a^R(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)}\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}_{a,0}^+(u)}$$

$$\mathbb{W}_a^B(u) = e^{-\frac{1}{2}L\mathcal{E}_a(u)}\mathbf{t}_{a,1}(u)$$



They come with a prescription for integrating the double poles

$$p_{ab}(u,v) \to p_{ab}(u+i0, v-i0)$$

Remark: transfer matrices in general are only defined up to gauge transformations

$$T_{a,s} \to g_1^{[a+s]} g_2^{[a-s]} g_3^{[s-a]} g_4^{[-a-s]} T_{a,s}$$

[Gromov,Kazakov,Leurent,Volin]
[Kazakov,Leurent,Volin]

In our case the gauge is fixed and the T-t's are normalised as in GKLV

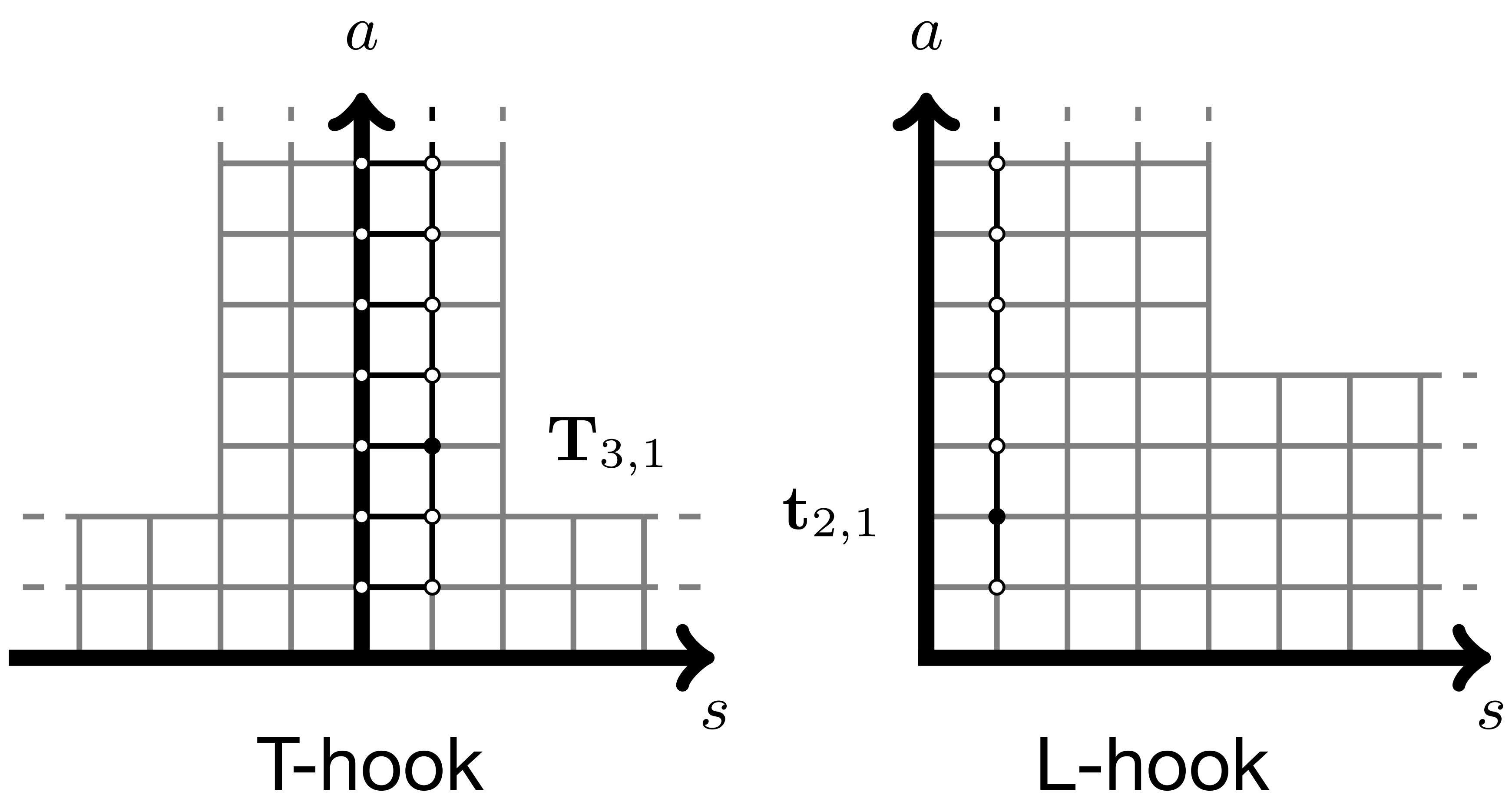
Conjecture

Exact weights

$$\mathbb{W}_a^L(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)}\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^-_{a,0}(u)}$$

$$\mathbb{W}_a^R(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)}\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^+_{a,0}(u)}$$

$$\mathbb{W}_a^B(u) = e^{-\frac{1}{2}L\mathcal{E}_a(u)}\mathbf{t}_{a,1}(u)$$



They come with a prescription for integrating the double poles

$$p_{ab}(u,v) \to p_{ab}(u+i0, v-i0)$$

Remark: transfer matrices can be computed efficiently using QSC and Q functions

[Gromov,Kazakov,Leurent,Volin]
[Kazakov,Leurent,Volin]

Last ingredient is the normalization factor $\mathcal{N}$ - it is believed to be given in terms of an infinite dimensional determinant associated with the TBA equations

Fishnets

Wheeled 3pt functions

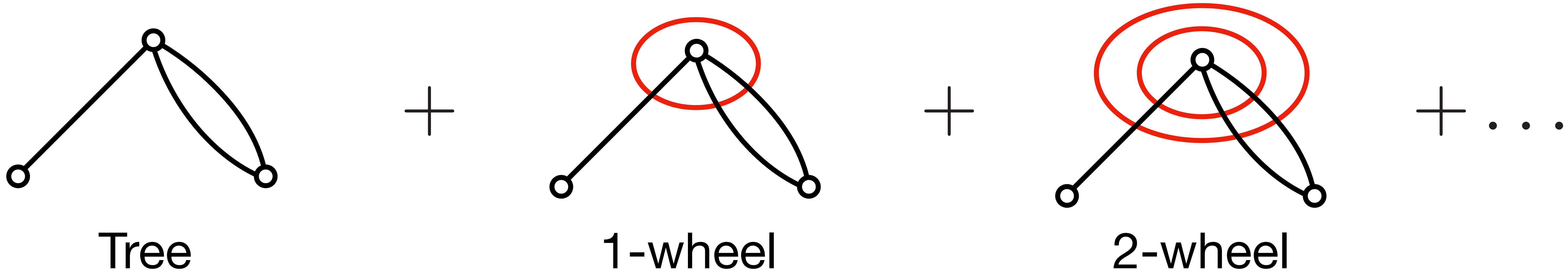
Conjecture for N = 4 three-point functions admit a “reduction” to the fishnet theory

One should consider structure constants for 2 open-string operators and 1 closed string operator

$$C_{123} = \langle Z_1^{J_1}(1)\,\mathcal{O}_{J,S}(0)\,Z_2^{J_2}(\infty)\rangle$$

Simplest case is when the closed string operator is the spin chain vacuum $\mathcal{O}_{J,S=0}(0) = \mathrm{Tr}\, Z^J$

These structure constants receive contributions from wheeled diagrams



Wheeled 3pt functions

Diagrammatic picture for hexagon sums: wheels = mirror magnons

Magnons are describing states flowing along the wheels (scalar fields and derivatives thereof)

Hexagon form factors can be constructed more rigorously using SoV techniques [Derkachov,Olivucci] [Olivucci]

Divergences of hexagon construction originate from UV divergences of the wheels [Gurdogan,Kazakov] [BB,Caetano,Fleury]

It may be possible to perform the “renormalization” rigorously

(In particular for the vacuum structure constants via OPE for 4pt functions of protected operators)

Wheeled 3pt functions

N = 4 conjecture simplifies in the fishnet theory

1) Less components to consider: bottom channel contribution is absent here

2) Reduced spectrum of magnons to match fields in the fishnet theory

3) Connection with XXX spin chain: meromorphic functions of rapidities (no cuts)

All we need to write conjectures are Y-functions and transfer matrices for fishnet T-hook

(should be constructible using Q functions as in N = 4 SYM)

Interesting open problem: Can we perform the sum over the bound states in this limit?

And make contact with SoV like representations

See Fedor's talk

Small spin limit

Motivation

Evaluating hexagon sums is remarkably difficult in general

State of the art:

5 loops at weak coupling (for ratio of structure constants) & classical limit at strong coupling

Simplifications occur if we move away from integer spins and expand around S = 0

Small spin limit behaves as a near BPS expansion

It proved useful in study of the spectral problem, notably to explore strong coupling regime

[BB][Gromov]
[Gromov,Valatka]
[Gromov,Levkovich-Maslyuk,Sizov,Valatka]

Advantage: many problems linearize in this limit and can be studied at any coupling

QSC at small spin

Focus on simplest family of states on leading Regge trajectory (minimal dimension $\Delta = \Delta_J(S)$)

QSC solution can be constructed very explicitly for any J in terms of Bessel functions

[Gromov,Levkovich-Maslyuk,Sizov,Valatka]

$$\mathbf{P}_1 = \mathbf{P}^4 = \epsilon x^{-J/2}, \qquad \mathbf{P}_2 = -\mathbf{P}^3 = -\epsilon x^{J/2} \sum_{n=J/2+1}^{\infty} I_{2n-1} x^{1-2n}$$

$$\mathbf{P}_3 = \mathbf{P}^2 = \epsilon \left(x^{-J/2} - x^{J/2} \right)$$

$$\mathbf{P}_4 = -\mathbf{P}^1 = \epsilon x^{J/2} \sum_{n=J/2+1}^{\infty} I_{2n-1} x^{1-2n} - \epsilon x^{-J/2} \sum_{n=1-J/2}^{\infty} I_{2n-1} x^{2n-1}$$

$$\epsilon^2 = \frac{2\pi i S}{J I_J(4\pi g)} \to 0$$

$I_k = I_k(4\pi g)$ = Bessel

This may be used to fix scaling dimension to all loops at small spin

[BB][Gromov]

$$\gamma = \Delta - S - J = \gamma_J^{(1)} S + \mathcal{O}\left(S^2\right), \qquad \gamma_J^{(1)} = \frac{4\pi g I_{J+1}(4\pi g)}{J I_J(4\pi g)}$$

One may also construct NLO solution and extract exact integral rep for so-called curvature function

Hexagons at small spin

Using this data one may obtain exact representations for components of the structure constants

Recall

$$C_{123} = \mathcal{N} \times \mathcal{A}(\ell_A) \times \mathcal{B}(\ell_B)$$

$$\ell_A = \frac{J_1 + J - J_2}{2}$$

$$\ell_B = \frac{J_1 + J_2 - J}{2}$$

Bottom (B) component is function of the transfer matrix

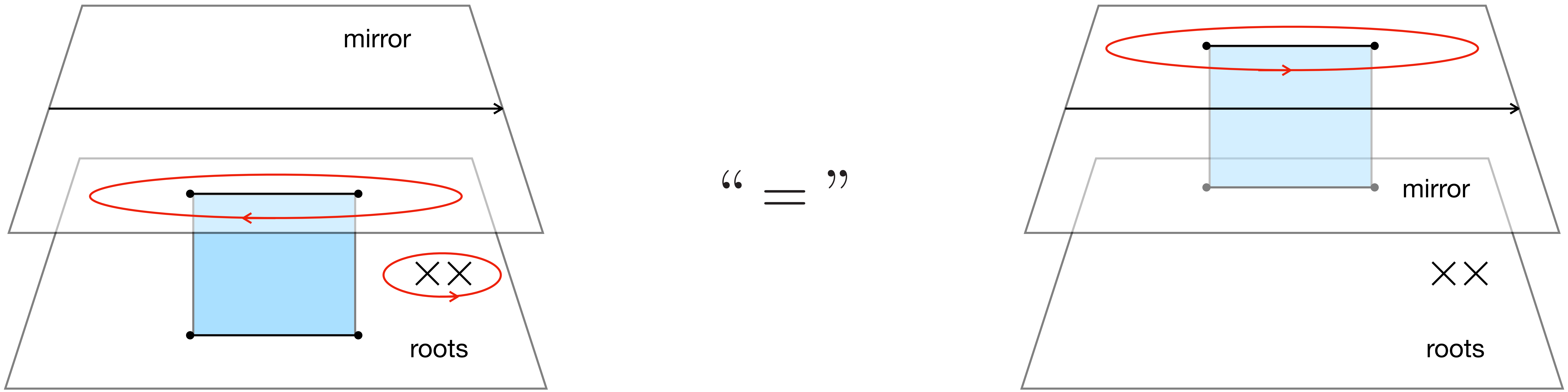
$$\mathbf{t}_{a,1}(u) = -\sum_{b=1}^{4} \mathbf{P}_b^{[+a]}(u)\widetilde{\mathbf{P}}^{b\,[-a]}(u) + \mathcal{O}\left(S^2\right)$$

A small transfer matrix implies that the mirror sum can be truncated to 1-magnon exchange

$$\mathcal{B} = 1 + \sum_{a=1}^{\infty} \int \frac{du}{2\pi} e^{-\frac{1}{2}(J_1+J_2)\mathcal{E}_a(u)} \mu_a(u)\, \mathbf{t}_{a,1}(u) + \mathcal{O}\left(S^2\right)$$

A-component through analytic continuation

The A component can also be cast in the same form through analytic continuation



Combine mirror and root contours together and deform to upper cut

Result is same as for B up to small modifications

Structure constants at small spin

The A component can also be cast in the same form through analytic continuation **[BB,Georgoudis]**

$$\mathcal{A} = 1 + S\,F_J(-\ell_A) + \mathcal{O}\left(S^2\right)\,, \qquad \mathcal{B} = 1 + S\,F_J(\ell_B) + \mathcal{O}\left(S^2\right)$$

Same function F for both A and B but different arguments

This function can be given to all loops as

$$F_J(\ell) = -\frac{ig}{2}\oint \frac{dxdy}{(2\pi)^2 xy}\frac{x-y}{xy-1}t_J(x,y)(\psi(1+iu-iv)-\psi(1)+(u\leftrightarrow v))$$

With t_J related to the transfer matrix (generating function of ratio of Bessel functions)

Representation is reminiscent of the one obtained for the curvature function

Unfortunately, it is not clear how to perform a similar expansion for the normalization (at finite coupling)

Explicit results

Ex. For shortest operators

$$F_{J=2}(-1) = -8g^2\zeta_3 + g^4(-32\zeta_2\zeta_3 + 90\zeta_5) + g^6(160\zeta_3\zeta_4 + 288\zeta_2\zeta_5 - 1120\zeta_7) + g^8(-1440\zeta_4\zeta_5 - 896\zeta_3\zeta_6 - 3360\zeta_2\zeta_7 + 14700\zeta_9) + \mathcal{O}\left(g^{10}\right)$$

General representation

$$F_J(\ell) = f_J(\ell) + f_J(-J-\ell)$$

with

$$f_J(\ell) = \sum_{k=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^{k+\ell+1}g^{2k+1}\Gamma(2k)\Gamma(2k+2)\,\zeta_{2k+1}\,\epsilon(n)\,\mathbf{I}_{2n-J-1}}{\Gamma(1+k+n)\Gamma(2+k-n)\Gamma(k+\ell+n)\Gamma(1+k-\ell-n)}$$

where $\epsilon(n) = -1$ for n negative and $\epsilon(n) = 1$ otherwise

and $$\mathbf{I}_n = \frac{2\pi I_n(4\pi g)}{J I_J(4\pi g)}$$

Strong coupling

Strong coupling

Significant progress has been made in studying structure constants at strong coupling wt string theory

Using reduction to flat-space string theory, dispersive sum rules and analyticity/sv

[Alday,Hansen]
[Alday,Hansen,Silva]
[Julius,Sokolova]

Small spin provides valuable data for fixing structure constants at strong coupling

Suitable ansatz allows one to connect this limit to regime short physical operators and classical strings

A string ansatz

Ansatz: $$\frac{C_{123}}{C^{(0)}_{123}} = \frac{\Gamma(AdS)}{\Gamma(Sphere)} \times \frac{\mathcal{D}}{\lambda^{S/4}\Gamma\left(1+\frac{S}{2}\right)}$$ with $$C^{(0)}_{123} = \frac{\sqrt{J_1 J_2 J}}{N}$$

Gamma factors: $$\Gamma(AdS) = \frac{\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta+S}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta_1+\Delta+S}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta+S}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_1+\Delta+S}{2}\right)}{\sqrt{\Gamma(\Delta+S)\Gamma(\Delta+S-1)}}$$

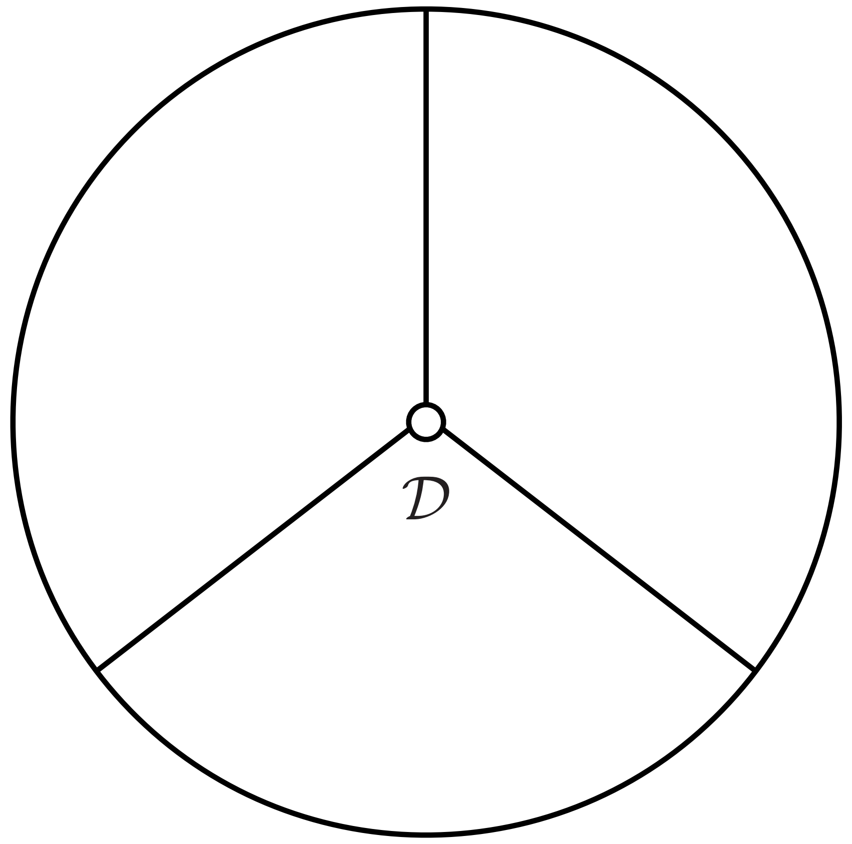
Similarly for the sphere with $S \to 0\,,\ \ \Delta \to J$

[Bargheer,Minahan,Pereira][Minahan,Pereira]
[Costa,Goncalves,Penedones][Alday,Hansen,Silva]

Insight comes from structure of 3pt functions in string theory

Gamma functions capture contribution from cubic Witten diagram

Rest is meant to capture flat-space string amplitude and its curvature corrections



Regularity assumptions

D-coefficient has a simpler expansion than the structure constant C [BB,Georgoudis]

Assumption 1: after taking the logarithm, it admits a strong coupling expansion

$$\log \mathcal{D} = D_1 + \frac{D_2}{\sqrt{\lambda}} + \frac{D_3}{\lambda} + \dots$$

with coefficient D_k that is a polynomial of degree k in the spin S and angular momenta J's

Assumption 2: coefficients interpolate smoothly between small spin and classical limit

They parallel observations made for the square of the scaling dimension [BB][Gromov,Valatka]

$$\Delta^2 = J^2 + \sqrt{\lambda}\, A_1 + A_2 + \frac{A_3}{\sqrt{\lambda}} + \dots$$

with A_k a polynomial of degree k in S and J

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Ex. For scaling dimension of twist-2 operators [BB][Gromov,Valatka]

$$\Delta^2 = 2\sqrt{\lambda}\, S + \left(4 - S + \frac{3}{2}S^2\right) + \frac{1}{\sqrt{\lambda}}\left(\frac{15}{4}S + \frac{3-24\zeta_3}{8}S^2 - \frac{3}{8}S^3\right) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

Evidence from two-loop string data

Recent two-loop string data for shortest operators support these assumptions

[Alday,Hansen]
[Alday,Hansen,Silva]
[Caron-Huot,Coronado,Zahraee]

After converting this data to the D-coefficient we find for $J_1 = J_2 = J = 2$

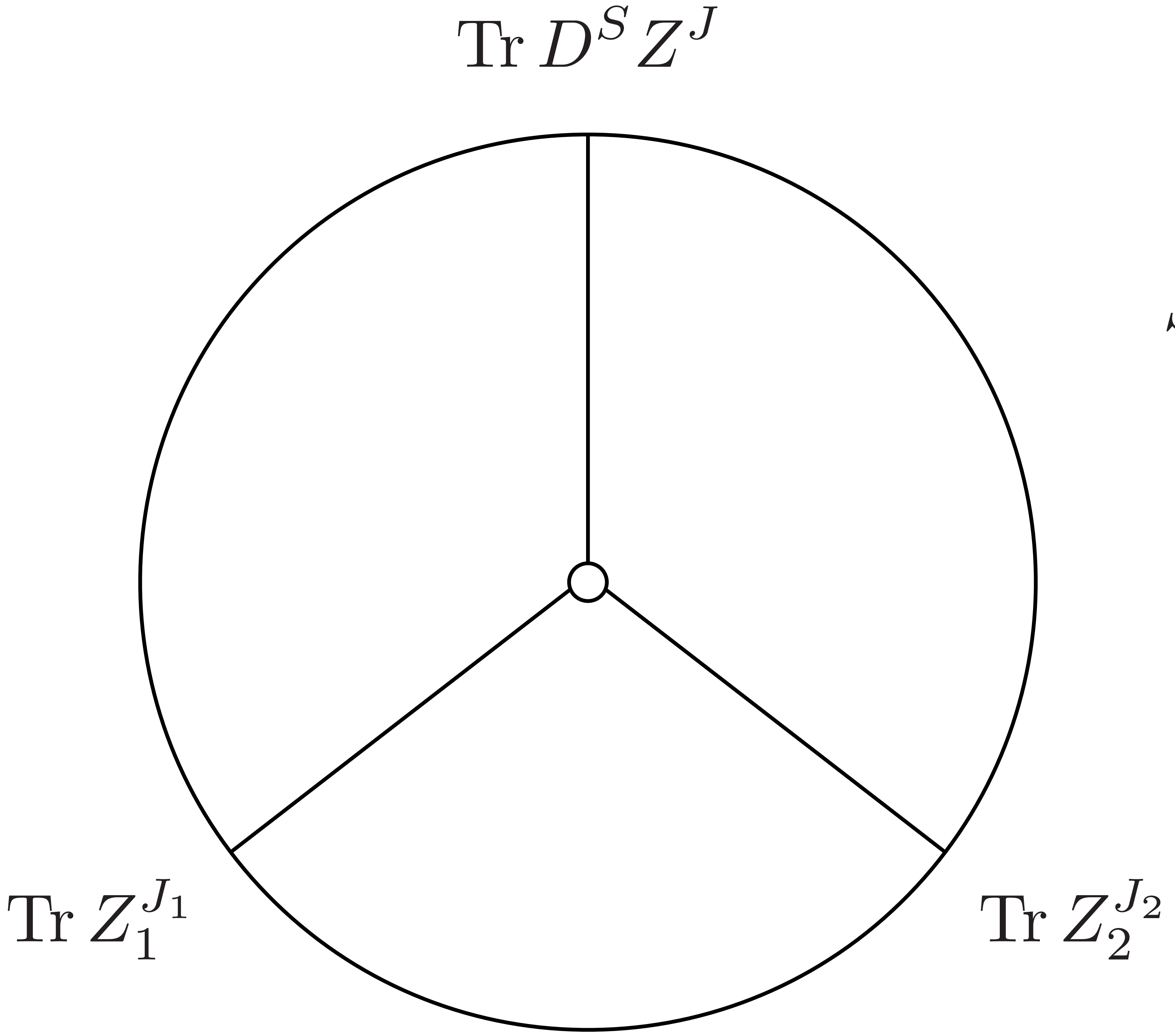
$$\log \mathcal{D}_{222} = \frac{1}{\sqrt{\lambda}} \left[\frac{5}{8} S - \frac{7 - 4\zeta_3}{16} S^2 \right] + \frac{1}{\lambda} \left[-\frac{13 + 24\zeta_3}{32} S - \frac{49 - 8\zeta_3}{64} S^2 + \frac{25 - 12\zeta_3 - 12\zeta_5}{64} S^3 \right] + \mathcal{O}\left(\frac{1}{\lambda^{3/2}} \right)$$

Each coefficient is a polynomial in S with degree = k = loop order +1

Consistent with our ansatz (for $D_1 = 0$)

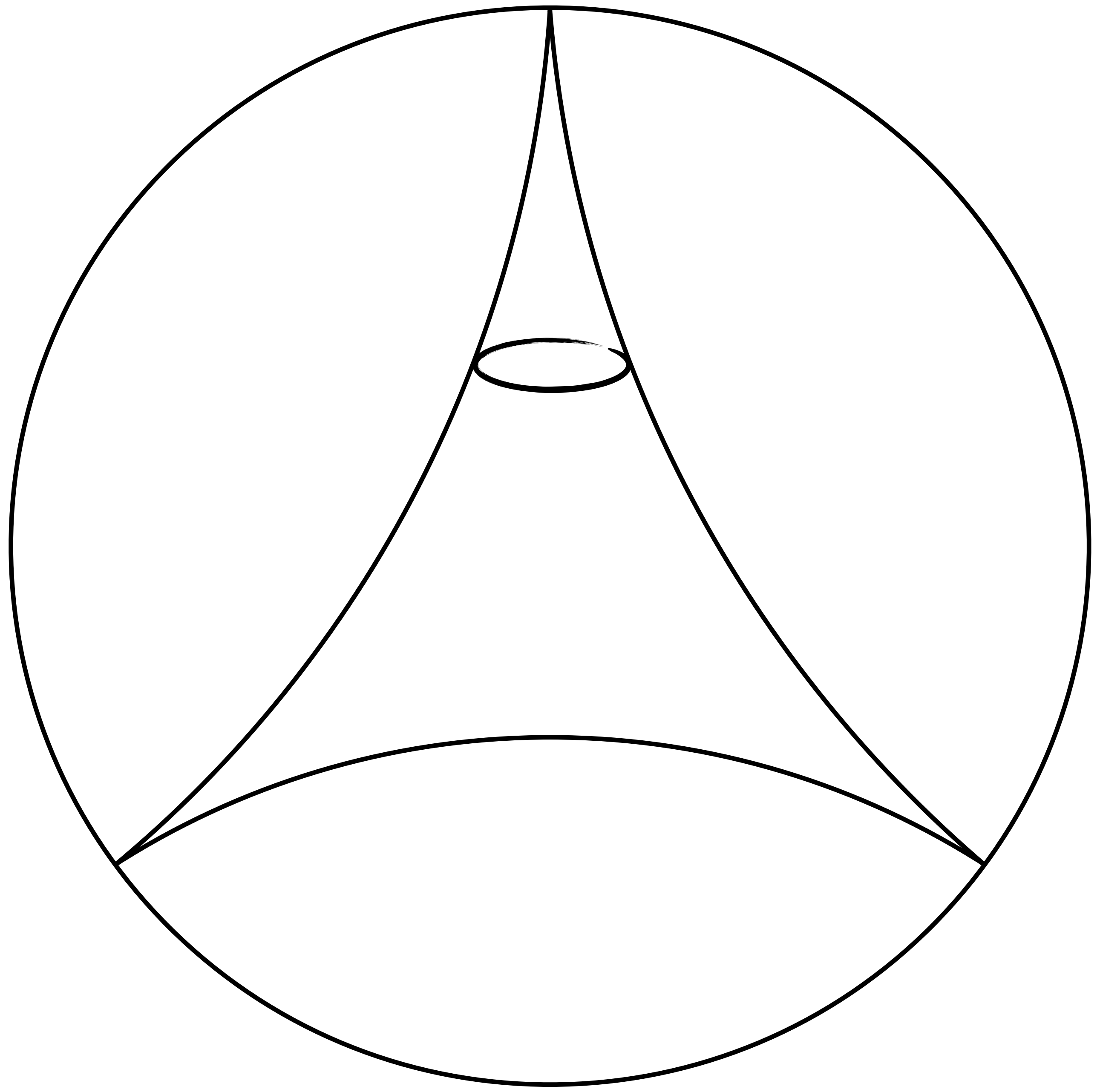
Further evidence comes from classical string (for terms of maximal degree $\sim S^{k+1}/(\sqrt{\lambda})^k$)

Classical analysis



$S, J, J_{1,2} \sim \sqrt{\lambda} \gg 1$

Classical spinning strings



Structure constant controlled by Area of minimal surface in AdS

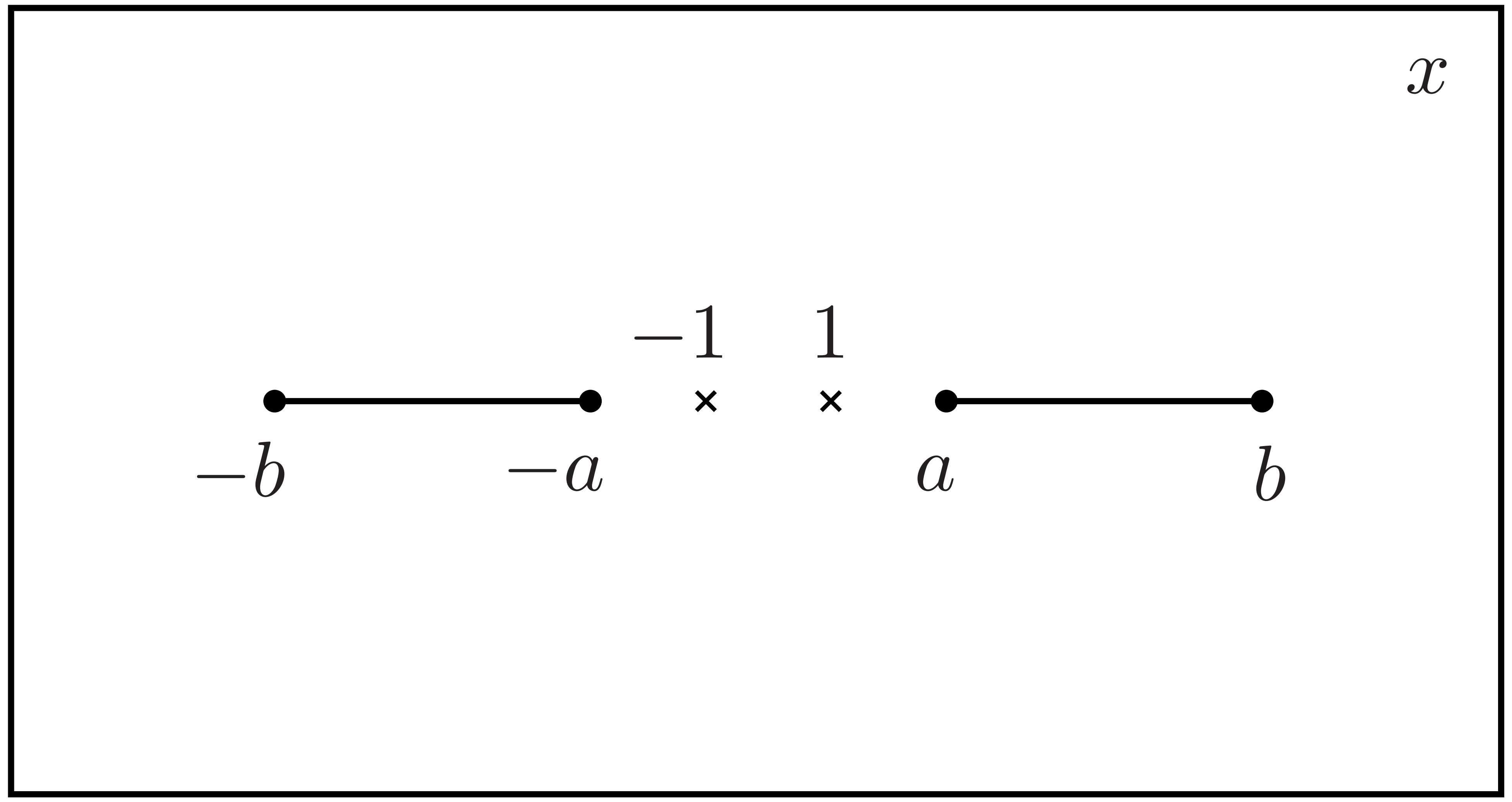
Classical string data

[Kazakov,Marshakov,Minahan,Zarembo]
[Kostov,Staudacher][Casteill,Kristjansen][Gromov]

Complete spectral data (solution) encoded in resolvent

$$R(x) = 2x \int_a^b \frac{dy\, V'(y)}{x^2 - y^2} \sqrt{\frac{(x^2-b^2)(x^2-a^2)}{(b^2-y^2)(y^2-a^2)}}$$

$$V'(x) = \mathrm{sgn}(x) - \frac{2\mathcal{J}x}{x^2-1}$$



Elliptic curve with parameters controlling angular momentum J, spin S and energy E of the string

$$\frac{J}{\sqrt{\lambda}} = \mathcal{J} = \frac{\sqrt{(a^2-1)(b^2-1)}}{\pi b} K\left(1 - \frac{a^2}{b^2}\right)$$

$$\frac{S}{\sqrt{\lambda}} = \mathcal{S} = \frac{ab+1}{2\pi ab}\left[b\, E\left(1-\frac{a^2}{b^2}\right) - a\, K\left(1-\frac{a^2}{b^2}\right)\right]$$

$$\frac{\Delta}{\sqrt{\lambda}} = \mathcal{E} = \frac{ab-1}{2\pi ab}\left[b\, E\left(1-\frac{a^2}{b^2}\right) + a\, K\left(1-\frac{a^2}{b^2}\right)\right]$$

Classical structure constant

[Kazama,Komatsu,Nishimura]

$$\sqrt{\lambda} \gg 1$$

Classica Area

$$\log C_{123} \sim \sqrt{\lambda} \times \left(\text{Area} = \mathcal{A}^{\text{cl}} + \mathcal{B}^{\text{cl}} + \mathcal{N}^{\text{cl}} \right)$$

Classical bridge lengths

$$\mathcal{L}_A = \frac{\mathcal{J}_1 - \mathcal{J}_2 + \mathcal{J}}{2}$$
$$\mathcal{L}_B = \frac{\mathcal{J}_1 + \mathcal{J}_2 - \mathcal{J}}{2}$$

Components:

$$\mathcal{A}^{\text{cl}} = \mathcal{A}^{\text{cl}}_{\text{asy}} + I_1[\mathcal{L}_A] + I_1[\mathcal{J} - \mathcal{L}_A]$$
$$\mathcal{B}^{\text{cl}} = I_{-1}[\mathcal{L}_B] + I_1[\mathcal{J} + \mathcal{L}_B]$$
$$\mathcal{N}^{\text{cl}} = \mathcal{N}^{\text{cl}}_{\text{asy}} - I_2[\mathcal{J}]$$

Mirror integral:

$$I_q[\mathcal{L}] = \int_{U^-} \frac{dx\,(x - 1/x)}{8\pi^2 x} \left[\text{Li}_2\left(e^{\frac{4\pi i \mathcal{L} x}{x^2-1} + iqR(x)} \right) + \text{Li}_2\left(e^{\frac{4\pi i \mathcal{L} x}{x^2-1} - iqR(1/x)} \right) \right]$$
$$- (\text{same with } R \to 0)\,,$$

Asymptotic integral:

[Gromov,Sever,Vieira]

$$\mathcal{N}^{\text{cl}}_{\text{asy}} = -\int_a^b \frac{dx(x - 1/x)}{4\pi^2 x} \left[\text{Li}_2\left(e^{-2\pi\rho(x)} \right) + \pi^2 \rho^2(x) - \zeta_2 \right],$$

Classical structure constant

Goal: plug elliptic solution in Area and compute integrals when $\mathcal{S}, \mathcal{J}, \mathcal{J}_1, \mathcal{J}_2 \to 0$

Tricky: Resolvent R & Integral I are singular in this limit

Analysis more tractable if **first** take small spin limit and **then** send lengths to zero

Similar to spectrum [Gromov,Valatka]

1) $\mathcal{S} \to 0$ (short cut limit, a <-> b)

2) $\mathcal{J}, \mathcal{J}_1, \mathcal{J}_2 \to 0$

We still get some terms that are singular in the end

But **magic** is that they all cancel out after dividing by ratio of Gamma functions for D-coefficient

Classical structure constant

Ex. At leading order at small spin

$$\mathcal{D}^{\rm cl} = \mathcal{D}^{\rm cl}_1\, \mathcal{S} + \mathcal{O}\left(\mathcal{S}^2\right)$$

where

$$\mathcal{D}^{\rm cl}_1 = -\frac{1}{4}\log\left(1+\mathcal{J}^2\right) + \sum_{k=1}^{\infty} \frac{\zeta_{2k+1}}{2k+1}\, c_k(\mathcal{J}) P_k(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J})$$

with P_k a homogeneous polynomial of deg 2k in the lengths

$$P_k = -2\mathcal{J}^{2k} + \frac{1}{2^{2k+1}\mathcal{J}} \sum_{\sigma_1,\sigma_2=\pm} \left(\mathcal{J} + \sigma_1 \mathcal{J}_1 + \sigma_2 \mathcal{J}_2\right)^{2k+1}$$

and

$$c_k(\mathcal{J}) = \sqrt{1+\mathcal{J}^2} \sum_{n=k}^{\infty} (-1)^n \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})\Gamma(1+n)} \mathcal{J}^{2(n-k)}$$

Importantly it is regular in all 3 lengths $\mathcal{J}, \mathcal{J}_1, \mathcal{J}_2$

Classical structure constant

Higher order terms display higher singularities but simplifications still observed for D-coefficients

Natural all-order decomposition $\mathcal{D}^{\rm cl} = \mathcal{D}^{\rm cl}_\rho + \mathcal{D}^{\rm cl}_\zeta$

Zeta free part controlled by density

$$\mathcal{D}^{\rm cl}_\rho = -\int_a^b \frac{(x-1/x)\,dx}{4\pi x}\rho(x)\log\left[\frac{(x^2-1)^2\rho^2(x)}{2e\mathcal{S}x^2}\right]$$

Easily checked (empirically) to be smooth in all variables

$$\mathcal{D}^{\rm cl}_\rho = -\frac{1}{4}\log\left(1+\mathcal{J}^2\right)\,\mathcal{S} - \frac{7+4\mathcal{J}^2}{16\left(1+\mathcal{J}^2\right)^{3/2}}\,\mathcal{S}^2 + \frac{150+120\mathcal{J}^2+29\mathcal{J}^4}{384\left(1+\mathcal{J}^2\right)^3}\,\mathcal{S}^3 - \frac{1785+1748\mathcal{J}^2+640\mathcal{J}^4+86\mathcal{J}^6}{3072\left(1+\mathcal{J}^2\right)^{9/2}}\,\mathcal{S}^4 + \mathcal{O}\left(\mathcal{S}^5\right)$$

Classical structure constant

Higher order terms display higher singularities but simplifications still observed for D-coefficients

Natural all-order decomposition $\mathcal{D}^{\rm cl} = \mathcal{D}^{\rm cl}_\rho + \mathcal{D}^{\rm cl}_\zeta$

Zeta full part controlled by resolvent

$$\mathcal{D}^{\rm cl}_\zeta = \sum_{\mathcal{L} \in L} Z\left[R_{\mathcal{L}}\right] - Z\left[2R_{\mathcal{J}/2}\right] \qquad L = \{\mathcal{L}_A, \mathcal{J} - \mathcal{L}_A, -\mathcal{L}_B, \mathcal{J} + \mathcal{L}_B\}$$

Where

$$Z\left[R_{\mathcal{L}}\right] = \int_{-i\infty}^{i\infty} \frac{(x+1/x)}{(2\pi)^2 i} \left[\frac{1}{2}\log\frac{\Gamma\left(1-\frac{R_{\mathcal{L}}(x)}{2\pi}\right)}{\Gamma\left(1+\frac{R_{\mathcal{L}}(x)}{2\pi}\right)} - \frac{\gamma_{\rm E} R_{\mathcal{L}}(x)}{2\pi}\right] dR_{\mathcal{L}}(x) - (R_{\mathcal{L}} \to \widehat{R}_{\mathcal{L}})$$

And $R_{\mathcal{L}}(x) = \frac{4\pi\mathcal{L}x}{x^2-1} + R(x)\,, \qquad \widehat{R}_{\mathcal{L}}(x) = \frac{4\pi\mathcal{L}x}{x^2-1}$ Much easier to expand at small R!

2-loop prediction

Polynomiality + small spin data + classical limit yield a prediction for 2-loop structure constant

$$
\begin{aligned}
&\log \mathcal{D}_{J_1 J_2 J} \\
&\quad = \frac{1}{\sqrt{\lambda}} \left[\frac{5}{8}\, S - \frac{7-4\zeta_3}{16}\, S^2 \right] \\
&\quad + \frac{1}{\lambda} \left[\frac{\big(19-8J^2\big) + 8\big(1+J^2-\vec{J}^2\big)\zeta_3}{32}\, S - \frac{49-8\zeta_3}{64}\, S^2 + \frac{25-12\zeta_3-12\zeta_5}{64}\, S^3 \right] \\
&\quad + \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right)
\end{aligned}
$$

Extend existing string data to operators of arbitrary lengths

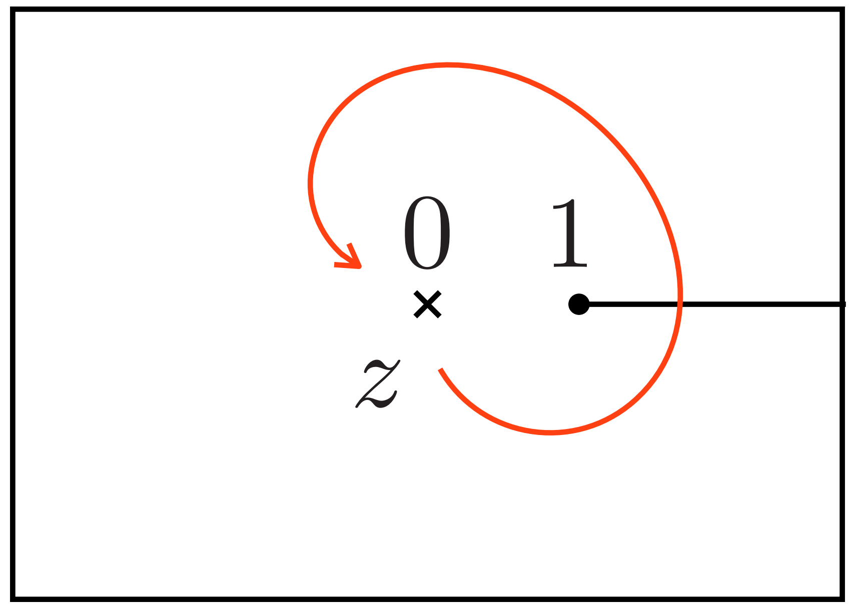
Regge limit

Consider a four-point function of chiral primary operators

$$\langle \mathrm{Tr}\, Z_1^2\, \mathrm{Tr}\, Z_2^2\, \mathrm{Tr}\, Z_3^p\, \mathrm{Tr}\, Z_4^p \rangle \propto \mathcal{G}_{22pp}(z,\bar{z})$$

Regge limit = short distance limit on 2nd sheet $\sigma \to 0$

$$z = \sigma e^{\rho}\,, \qquad \bar{z} = \sigma e^{-\rho}$$



Factorization formula

[Costa,Gonçalves,Penedones]
[Costa,Drummond,Gonçalves,Penedones]

$$\mathcal{G}_{22pp} = \frac{\pi^2 i\sqrt{\lambda}}{Z(p)} \int_{-\infty}^{\infty} \frac{S' d\nu}{\nu} \frac{\gamma_S(\nu)\gamma_S(-\nu)\Omega_{i\nu}(\rho)}{(\sqrt{\lambda}\,\sigma)^{1+S}} \frac{e^{\frac{i\pi S}{2}}\Gamma\left(-\frac{S}{2}\right)}{\Gamma\left(1+\frac{S}{2}\right)} \mathcal{D}_{222}\mathcal{D}_{pp2}$$

Here S = spin as function of scaling dimension $\Delta = i\nu$

$$S = -\frac{\nu^2+4}{2\sqrt{\lambda}}\left(1+\frac{1}{2\sqrt{\lambda}}\right) + \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right)$$

Spin is naturally small in this regime!

Remark appearance of scale $\xi = \sqrt{\lambda}\,\sigma$

Pomeron dominates when $\xi \ll 1$ otherwise all trajectories contribute

Outlook

Hexagons are useful tools to explore structure constants in various regimes as well as the connections with QSC in SYM and fishnet theory

Re-summation techniques must be designed to perform mirror sums

Small spin hexagon provides valuable data for fixing structure constants at strong coupling

Hard to be rigorous here - a polynomial ansatz is needed to relate this data to string theory

Polynomiality also proved useful for studying structure constants of sub-leading trajectories

[Julius,Sokolova]

Interesting to understand general structure at strong coupling for all these trajectories and if connections to hexagon representation can be established