Graph-Building Operators for Fishnet Integrals

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The planar fishnet theory and its generalisations admit very few Feynman diagrams, which have a regular (square-lattice) structure. This can be used to

- solve these theories,
- understand the origin of their integrability,
- understand the origin of their holographic dual,
- gain insight into richer theories, e.g. $\mathcal{N} = 4$ SYM,
- (in)directly compute individual Feynman integrals.

The Fishnet Theory

$$\mathcal{L} = N_c \,\mathrm{Tr}\left[X^\dagger \left(-\partial_\mu \partial^\mu\right)^\delta X + Z^\dagger \left(-\partial_\mu \partial^\mu\right)^{\tilde{\delta}} Z - (4\pi)^{\frac{d}{2}} \xi^2 Z^\dagger X^\dagger Z X\right] ,$$

$$\delta + \tilde{\delta} = \frac{d}{2} , \qquad\qquad 0 < \delta < \frac{d}{2} .$$

[Gürdoğan and Kazakov (2015)] [Kazakov and Olivucci (2018)]

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Feynman rules:

$\propto \dfrac{1}{(x-y)^{2\tilde{\delta}}}$

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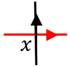


$\propto \xi^2 \int d^d x$

We work in the planar limit $N_c \to +\infty$.

Outline

Integrability through Graph-Building Operators

From Fishnet to $\mathcal{N} = 4$ SYM

- A Short Operator: $\mathrm{Tr}(FZ)$
- Mixing Between Operators and Between Scaling Limits

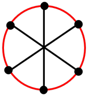
Mirror Channel and Separation of Variables

One-Dimensional Integrals

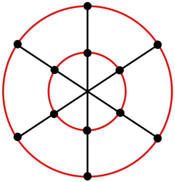
Integrability through Graph-Building Operators

Graph-Building Operators

$$\left\langle Tr(Z^J(x))\, Tr(Z^{\dagger J}(y)) \right\rangle \quad \leftrightarrow \quad 1 + \xi^{2J}$$



$$+ \; \xi^{4J}$$



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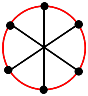

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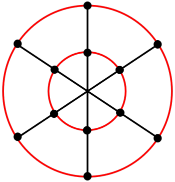
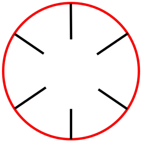
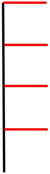


$$+ \cdots$$

Two graph-building operators: direct channel v mirror channel

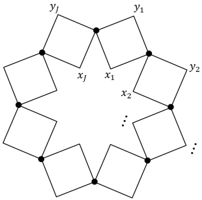




In principle, the integrals can be computed through the diagonalisation of either of these operators.

Transfer Matrix = Direct-Channel Graph-Building Operator

$$\mathrm{T}_J^{(\Delta_0)}(u) = \mathrm{Tr}_0\left(R_{01}(u)\cdots R_{0J}(u)\right) \qquad \longleftrightarrow$$



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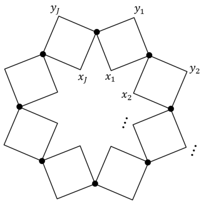
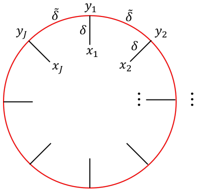


Yang–Baxter for the R-matrix $\Longrightarrow [\mathrm{T}_J^{(\Delta_0)}(u), \mathrm{T}_J^{(\Delta_0')}(v)] = 0.$

Taking $\Delta_0 = \delta$ and $\Delta_1 = \cdots = \Delta_J = \tilde{\delta}$, one observes that

$$\epsilon^J \mathrm{T}_J^{(\Delta_0)} \left(-\frac{d}{4} + \epsilon \right) \propto_{\epsilon \to 0} \widehat{H}_J \qquad \underset{\epsilon\to 0}{\longleftrightarrow}$$



[Gürdoğan and Kazakov (2015)][Gromov, Kazakov, Korchemsky, Negro, and Sizov (2017)]

The 2-point function is essentially reduced to the computation of

$$\sum_{M=0}^{+\infty} \xi_1^{2MJ} \widehat{H}_J^M = \frac{1}{1-\xi_1^{2J}\widehat{H}_J}\,.$$

Eigenvectors of $\widehat{H}_J$ with eigenvalue $E=\xi^{-2J}$ represent primary operators of the fishnet theory (and their descendants). This is given by the representation of the conformal group $(\Delta(\xi^2), \ell, \bar{\ell})$ under which the eigenvector tranforms.

When $J = 2$, one can compute exactly

$$\left\langle \mathrm{Tr}[Z(x_1)Z(x_2)]\,\mathrm{Tr}\left[Z^\dagger(x_3)Z^\dagger(x_4)\right]\right\rangle$$

and extract from it (exact) conformal dimensions and OPE coefficients. Physical states correspond to symmetric traceless tensors of arbitrary rank $\ell \geqslant 0$; their dimensions are

$$\Delta_{\ell,\pm} = 2 + \sqrt{(\ell+1)^2 + 1 \pm 2\sqrt{(\ell+1)^2 + 4\xi^4}}\,.$$

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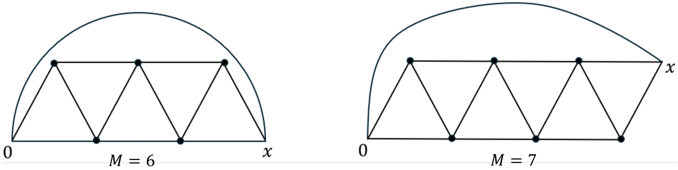
For arbitrary J, not completely solved.

Application: Periods of Zigzag Graphs



$$\frac{1}{|x|^4} \times \frac{4}{M} \binom{2M-2}{M-1} \left[1 - 2^{3-2M}(1-(-1)^M)\right] \zeta_{2M-3}$$

Conjectured in 1995 [Broadhurst and Kreimer (1995)]

Two proofs:

- single-valued multiple polylogarithms, graphical functions [Brown and Schnetz (2012)]
- $J = 2$ graph-building operator [Gromov, Kazakov, and Korchemsky (2018)] [Derkachov, Isaev, and Shumilov (2022)]

From Fishnet to $\mathcal{N} = 4$ SYM

Comments and Shortcomings of the Fishnet Theory

- The previous results are exact. In particular, for $J = 2$ and $\ell = 0$,

$$\Delta_{0,-} = 2 + \sqrt{2 - 2\sqrt{1 + 4\xi^4}} = 2 \pm 2\,\mathrm{i}\,\xi^2 + O(\xi^4)$$

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- On the other hand, $\Delta_{0,+}$ is the dimension of $Tr(Z\Box Z) + \ldots$ which we do not know exactly because there is mixing.

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- The fishnet theory is a logarithmic CFT: the dilatation operator is not diagonalisable.
- Neither fermions nor gauge boson in the fishnet theory.

[Gürdoğan and Kazakov (2015)] [Kade and Staudacher (2024)]

How can one incorporate back these protected or logarithmic operators?

New Double-Scaling Limits

[Ferrando, Sever, Sharon, and Urisman (2023)]

Operator-dependent limit:

$$e^{-i\gamma_3} \to \infty\,, \quad g \to 0\,, \quad \xi_n^2 = \frac{g^2\, e^{-i\frac{\gamma_3}{n}}}{8\pi^2} \quad \text{fixed}$$

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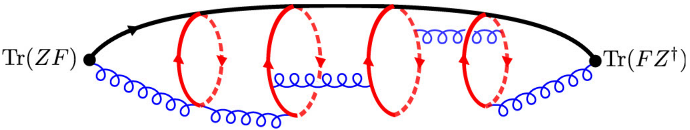
Example: for $\mathrm{Tr}(ZF)$, one must take $n = 2$ and the only diagrams that remain are



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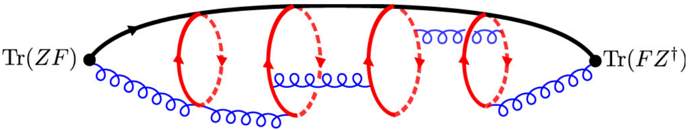


Following the procedure outlined previously, we find that

$$\Delta_{\mathrm{Tr}(FZ)} \underset{g\to 0\,,\xi_2\ \text{fixed}}{\longrightarrow} 2 + \sqrt{5 - 4\sqrt{1+\xi_2^4}}\,.$$

General Situation: Mixing

If we turn to longer operators, such as $\mathrm{Tr}\left(FZ^J\right)$ for $J > 1$, then $n = 1 + 1/J$.

But there is some form of mixing with $\mathrm{Tr}\left(XX^\dagger Z^J\right)$ (same double-scaling limit) and $\mathrm{Tr}\left(Z^J\right)$ (fishnet limit).

The relevant graph-building operator is a 3×3 matrix. We will show that it is integrable.

A Short Operator: $\mathrm{Tr}(FZ)$

Feynman Diagrams

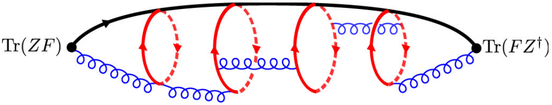
Double-scaling limit:

$$e^{-i\gamma_3} \to \infty\,, \quad g \to 0\,, \quad \xi_2^2 = \frac{g^2\, e^{-i\frac{\gamma_3}{2}}}{64\pi^4} \quad \text{fixed}\,.$$

Relevant interactions:

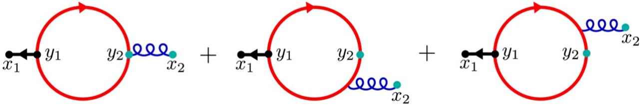
$$-\,i\,N_c g\,\mathrm{Tr}\big(\partial_\mu X^\dagger [A^\mu, X] + \partial_\mu X [A^\mu, X^\dagger]\big)\,,$$
$$2N_c g^2\,\mathrm{Tr}\big(X^\dagger A_\mu X A^\mu\big)\,, \quad \text{and} \quad 2N_c g^2\, e^{-i\gamma_3}\,\mathrm{Tr}\big(X^\dagger Z^\dagger X Z\big)\,.$$

Typical diagram:



Graph-Building Operator

$\widehat{H}_A$ depends on the gauge:



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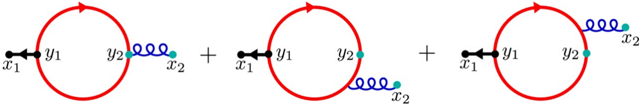

However, there exists a gauge-independent operator $\widehat{H}_F$ acting on antisymmetric tensors $\Psi_F^{\mu\nu}$ and such that: if $\Psi_F^{\mu\nu} = \partial_2^\mu \Psi_A^\nu - \partial_2^\nu \Psi_A^\mu$, then

$$\left[\widehat{H}_F \Psi_F\right]^{\mu\nu} = \partial_2^\mu \left[\widehat{H}_A \Psi_A\right]^\nu - \partial_2^\nu \left[\widehat{H}_A \Psi_A\right]^\mu .$$

$\Longrightarrow \langle \mathrm{Tr}(ZF)(x)\, \mathrm{Tr}(Z^\dagger F)(y) \rangle$ is gauge-independent in the double-scaling limit.

One can invert $\widehat{H}_F$:

$$\left[\widehat{H}_F^{-1}\Psi_F\right]^{\mu\nu} = \frac{1}{16}\left(\partial_2^\mu x_{12}^4 \Box_1 \partial_2^\rho \Psi_{F,\rho}{}^\nu - (\mu \leftrightarrow \nu)\right) .$$

Eigenvectors are fixed by the conformal covariance of the operator: three-point functions involving a scalar of dimension 1 and a rank-2 antisymmetric tensor of dimension 2.

Spectrum:

- $(\Delta_{\ell,\pm}, \ell, \ell)$ for $\ell \geqslant 1$ with

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The dimension of $\mathrm{Tr}(ZF)$ is $\Delta'_{0,-}$.

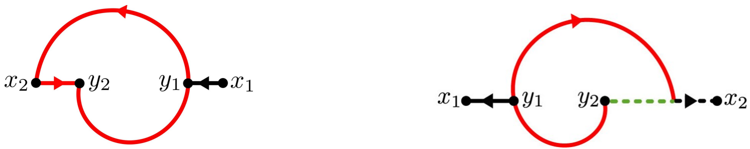
Other Short Operators

We performed a similar analysis for the following operators:

$$\text{Tr}\left(XX^{\dagger}Z\right) \quad \text{and} \quad \text{Tr}\left(X^{\dagger}XZ\right) \Longrightarrow n = 2$$

$$\text{Tr}(\psi_4 Z) \quad \text{or} \quad \text{Tr}\left(\psi_1^{\dagger}Z\right) \Longrightarrow n = \frac{4}{3}$$

$$\text{Tr}(\psi_2 Z) \quad \text{or} \quad \text{Tr}\left(\psi_3^{\dagger}Z\right) \Longrightarrow n = 4$$



Mixing Between Operators and Between Scaling Limits

Fishnet Contributions

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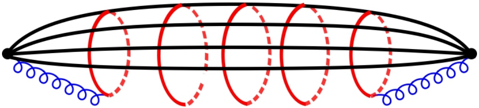
Let us consider the 2-pt function $\big\langle \mathrm{Tr}\big(Z^J F\big)(x)\,\mathrm{Tr}\big((Z^\dagger)^J F\big)(y)\big\rangle$. When $e^{-\mathrm{i}\gamma_3} \to +\infty$, the dominant contributions are



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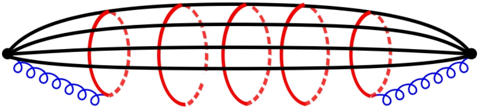
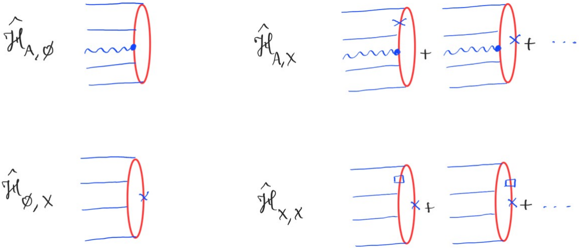


But $\mathrm{Tr}(Z^J F)$ is absent from the fishnet theory, so more graphs need to be taken into account.

Mixing

There is still an iterative structure: the graph-building operator is actually a matrix $\widehat{\mathcal{H}}$ with one row (and one column) for each state that participate in the mixing.

In our case, there are 3 intermediate states: $\mathrm{Tr}(Z^J)$, $\mathrm{Tr}(Z^J F)$ and $\mathrm{Tr}(Z^J X X^\dagger)$.



$\widehat{\mathcal{H}}$ is defined such that 2-point functions are essentially matrix elements of $\frac{1}{1-\widehat{\mathcal{H}}}$

Example:

$$\Big\langle \mathrm{Tr}(A^\mu(x_0)Z(x_1)\ldots Z(x_J))\,\mathrm{Tr}\big(Z^\dagger(z_J)\ldots Z^\dagger(z_1)\big)\Big\rangle$$
$$= -\frac{\mathrm{i}}{2}\int \frac{\langle x_0, x_1, \ldots, x_J|\left(\frac{1}{1-\widehat{\mathcal{H}}}\right)^\mu_{A\emptyset}|y_1,\ldots,y_J\rangle}{(4\pi^2)^J\prod_{i=1}^J (y_i - z_i)^2}\,\frac{\prod_{i=1}^J \mathrm{d}^4 y_i}{\pi^{2J}}\,.$$

The problem is still to diagonalise $\widehat{\mathcal{H}}$, and physical states correspond to those with eigenvalue equal to 1.

Double-Scaling Limit

$$e^{-\mathrm{i}\,\gamma_3} \to \infty\,, \quad g \to 0\,, \quad \xi_{1+1/J}^2 = \frac{g^2\, e^{-\mathrm{i}\,\frac{J}{J+1}\gamma_3}}{8\pi^2} \quad \text{fixed}$$

Each matrix element scales differently:

$$\widehat{\mathcal{H}} = \xi_{1+1/J}^{2(J+1)} \begin{pmatrix} g^{-2}\widehat{\mathcal{H}}_{\emptyset\emptyset} & g^{-1}\widehat{\mathcal{H}}_{\emptyset A} & g^{-1}\widehat{\mathcal{H}}_{\emptyset X} \\ g^{-1}\widehat{\mathcal{H}}_{A\emptyset} & \widehat{\mathcal{H}}_{AA} & \widehat{\mathcal{H}}_{AX} \\ g^{-1}\widehat{\mathcal{H}}_{X\emptyset} & \widehat{\mathcal{H}}_{XA} & \widehat{\mathcal{H}}_{XX} \end{pmatrix}\,.$$

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Some eigenvalues will diverge, some will go to zero. We focus on those which remain finite:

$$\widehat{\mathcal{H}}\Psi = E\Psi\,, \quad \text{with} \quad E = E_0 + O(g)\,, \quad E_0 \neq 0\,.$$

Effective Spectral Problem

At leading order, only the above 3×3 submatrix is relevant. Writing

$$\Psi = \begin{pmatrix} \Psi_{\emptyset,0}(x_1,\ldots,x_J) \\ \Psi^{\mu}_{A,0}(x_0,x_1,\ldots,x_J) \\ \Psi_{X,0}(x_0,x_1,\ldots,x_J) \end{pmatrix} + O(g)\,,$$

we get $\Psi_{\emptyset,0} = 0$ and

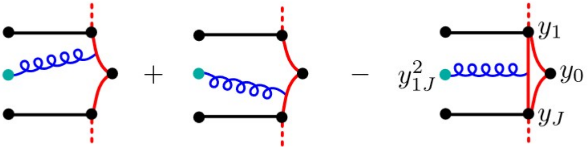
$$\xi_{1+1/J}^{2(J+1)}\widehat{\mathfrak{H}} \begin{pmatrix} \Psi_{F,0} \\ \Psi_{X,0} \end{pmatrix} = E_0 \begin{pmatrix} \Psi_{F,0} \\ \Psi_{X,0} \end{pmatrix}$$

for $\Psi^{\mu\nu}_{F,0} = \partial_0^{\mu}\Psi^{\nu}_{A,0} - \partial_0^{\nu}\Psi^{\mu}_{A,0}$, and some 2×2 matrix $\widehat{\mathfrak{H}}$ depending on all 9 matrix elements of $\widehat{\mathcal{H}}$.

$$\widehat{\mathfrak{H}}_{ij} = \widehat{\mathcal{H}}_{ij} - \widehat{\mathcal{H}}_{i\emptyset}\widehat{\mathcal{H}}_{\emptyset\emptyset}\widehat{\mathcal{H}}_{\emptyset j}\,, \quad (i,j) \in \{A, X\}$$

is a complicated matrix of integral operators but it is local and gauge invariant (contrary to $\widehat{\mathcal{H}}$). As an example:

$$\left[\widehat{\mathfrak{H}}_{FX}\Psi_X\right](\theta, x_0, \ldots, x_J) = \frac{\theta\cdot\partial_0}{4}\int\prod_{j=0}^{J}\frac{\mathrm{d}^4 y_j}{\pi^2}\frac{\Psi_X(y_0,\ldots,y_J)}{\prod_{i=0}^{J}(x_i-y_i)^2 y_{i,i+1}^2}$$
$$\times\int\frac{\mathrm{d}^4 z}{\pi^2}\frac{(x_0-y_0)^2}{(x_0-z)^2}\left[\frac{1}{2(z-y_1)^2(z-y_J)^2}\theta\cdot\partial_z\left(\frac{y_{10}^2(z-y_J)^2}{(z-y_0)^2}-\frac{y_{J0}^2(z-y_1)^2}{(z-y_0)^2}\right)\right.$$
$$\left.+\left(\frac{\theta\cdot(y_J-z)}{(y_J-z)^2}-\frac{\theta\cdot(y_1-z)}{(y_1-z)^2}\right)\left(\frac{y_{10}^2}{(z-y_1)^2(z-y_0)^2}+\frac{y_{J0}^2}{(z-y_J)^2(z-y_0)^2}-\frac{y_{1J}^2}{(z-y_1)^2(z-y_J)^2}\right)\right],$$



Surprisingly, $\widehat{\mathfrak{H}}$ has a relatively simple inverse:

$$\widehat{\mathfrak{H}}^{-1} = \begin{pmatrix} \theta\cdot\partial_0\, x_{J0}^2 x_{10}^2\,\partial_0\cdot\partial^{(\theta)} & 2\,\theta\cdot\partial_0\left(\frac{\theta\cdot x_{J0}}{x_{J0}^2}-\frac{\theta\cdot x_{10}}{x_{10}^2}\right)x_{J0}^2x_{10}^2 \\ 2\left(\frac{x_{10}\cdot\partial^{(\theta)}}{x_{10}^2}-\frac{x_{J0}\cdot\partial^{(\theta)}}{x_{J0}^2}\right)x_{J0}^2x_{10}^2\,\partial_0\cdot\partial^{(\theta)} & \partial_{0,\mu}\,x_{J0}^2x_{10}^2\,\partial_0^\mu + 8\,x_{10}\cdot x_{J0} \end{pmatrix}$$
$$\times\ \frac{\prod_{i=1}^{J-1} x_{i,i+1}^2 \prod_{i=1}^{J}\Box_i}{(-4)^{J+1}}\,,$$

where θ^μ is a polarisation vector such that $\{\theta^\mu,\theta^\nu\}=0$. It encodes the tensor structure: $\Psi^{\mu\nu}\mapsto\Psi=\theta^\mu\theta^\nu\Psi_{\mu\nu}$.

Integrability

We can construct a transfer matrix

$$T(u) = \mathrm{tr}_6 \left(L_{Y_0}^{(\rho_0)}(u) L_{Y_1}^{(1,0,0)}(u) \cdots L_{Y_J}^{(1,0,0)}(u) \right)$$

such that

$$T(0) = (-1)^{J+1} \widehat{\mathfrak{H}}^{-1} \, .$$

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We have checked that the 6×6 Lax matrices are solution to the RLL equation

$$R_{12}(u-v) L_{Y,1}^{(\rho_0)}(u) L_{Y,2}^{(\rho_0)}(v) = L_{Y,2}^{(\rho_0)}(v) L_{Y,1}^{(\rho_0)}(u) R_{12}(u-v) \,,$$

where $R_{12}(u)$ is the usual $O(5,1)$-invariant R-matrix.

[Zamolodchikov and Zamolodchikov (1979)]

The Lax matrices for sites $1, \ldots, J$ are the usual ones for scalar representations:

$$L^{(1,0,0)}_{Y,MN}(u) = u^2 \eta_{MN} - u\left(Y_M \partial_{Y^N} - Y_N \partial_{Y^M}\right) - \frac{1}{2} Y_M Y_N \Box_Y \,.$$

Embedding space: $1 \leqslant M \leqslant 6$, metric $\eta^{MN} = \mathrm{diag}(1,1,1,1,1,-1)$, and $Y^M Y_M = 0$.

But the representation at site 0 is reducible

$$\rho_0 = \underbrace{[\Theta \cdot \partial_Y (1,1,1)]}_{F} \oplus \underbrace{(2,0,0)}_{XX^\dagger}$$

and the Lax matrix appears to be new:

$$L^{(\rho_0)}_{Y,MN}(u) = u^2 \eta_{MN} - u\, q^{(\rho_0)}_{MN} + \mathcal{L}_{Y,MN}\,,$$

where the conformal generators are

$$q^{(\rho_0)}_{MN} = \begin{pmatrix} Y_M \partial_{Y^N} - Y_N \partial_{Y^M} + \Theta_M \partial_{\Theta^N} - \Theta_N \partial_{\Theta^M} & 0 \\ 0 & Y_M \partial_{Y^N} - Y_N \partial_{Y^M} \end{pmatrix}$$

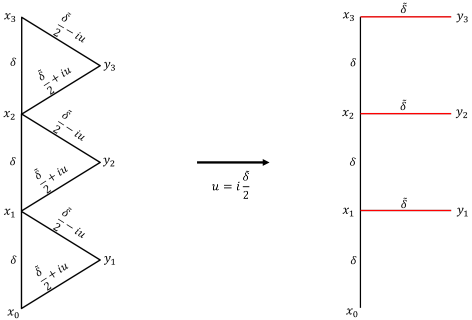
and the operator $\mathcal{L}_Y$ is

$$\mathcal{L}^{MN}_Y = -\frac{1}{2} \begin{pmatrix} (\Theta \cdot \partial_Y)\, Y^M Y^N (\partial_Y \cdot \partial_\Theta) & (\Theta \cdot \partial_Y) \left[Y^M \Theta^N - Y^N \Theta^M \right] \\ \left[Y^N \partial^M_\Theta - Y^M \partial^N_\Theta \right] (\partial_Y \cdot \partial_\Theta) & \frac{1}{2} \left[Y^M \Box_Y Y^N + Y^N \Box_Y Y^M \right] + 2\eta^{MN} \end{pmatrix} .$$

Mirror Channel and Separation of Variables

Diagonalisation of the Mirror Graph-Building Operator

Λ_N is part of a commuting family of open-spin-chain operators. The kernels of these integral operators are



Clear (spin-chain) interpretation in dimension 2: the eigenvectors form a basis of separated variables.

[Derkachov, Korchemsky, and Manashov (2002)] [Derkachov and Manashov (2014)]

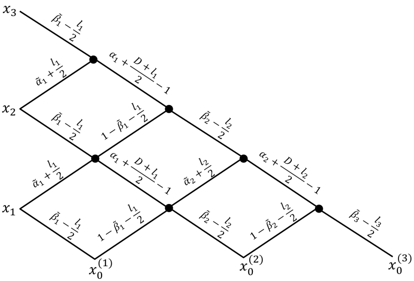
If $\delta \in \mathrm{i}\,\mathbb{R}$, then the operators act on $L^2(\mathbb{R}^d)^{\otimes N}$. We will perform the analytic continutation to $0 < \delta < \frac{d}{2}$ at the end.

When $N = 1$, the eigenvectors $|\vec{u}, C\rangle$ are parametrised by $\vec{u} = (u, l) \in \mathbb{R} \times \mathbb{N}$ and C, a symmetric traceless tensor of rank l:

$$\langle x|\vec{u}, C\rangle = \frac{C^{\mu_1 \ldots \mu_l} x_{\mu_1} \ldots x_{\mu_l}}{x^{2\left(\frac{d}{4}+\frac{\delta}{2}-\mathrm{i}\,u\right)}}$$

$$\Lambda_1 \,|\vec{u}, C\rangle = \underbrace{\frac{\Gamma(\delta)\Gamma\left(\frac{d}{4}+\frac{l-\delta}{2}+\mathrm{i}\,u\right)\Gamma\left(\frac{d}{4}+\frac{l-\delta}{2}-\mathrm{i}\,u\right)}{\Gamma(\tilde{\delta})\Gamma\left(\frac{d}{4}+\frac{l+\delta}{2}+\mathrm{i}\,u\right)\Gamma\left(\frac{d}{4}+\frac{l+\delta}{2}-\mathrm{i}\,u\right)}}_{\lambda_l(u)} |\vec{u}, C\rangle$$

For higher N, the eigenvectors are parametrised similarly, $|\vec{u}_1, \ldots, \vec{u}_N; \overbrace{C_1 \otimes \cdots \otimes C_N}^{C}\rangle$, and the eigenvalues are simply $\prod_{k=1}^{N} \lambda_{l_k}(u_k)$.

$C(\partial_0^{(1)}, \partial_0^{(2)}, \partial_0^{(3)})$ [] $\Big|_{x_0^{(1)} = x_0^{(2)} = x_0^{(3)} = 0}$

Symmetry

For any permutation τ,

$$|\vec{u}_1, \ldots, \vec{u}_N; C\rangle = \left|\vec{u}_{\tau^{-1}(1)}, \ldots, \vec{u}_{\tau^{-1}(N)}; \mathbb{S}(\vec{u}_1, \ldots, \vec{u}_N; \tau) C\right\rangle \ ,$$

where the S-matrix factorises into products of $2 \to 2$ scattering governed (up to a phase) by the relevant $O(d, \mathbb{C})$-invariant R-matrix $\mathbb{R}_{l,l'}(u - u')$ (solution to Yang–Baxter).

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This required new, explicit representations for the R-matrices for arbitrary spins. For example,

$$\begin{aligned} &\left[\mathbb{R}_{l_1,l_2}(u) C_1 \otimes C_2\right](x, y) \\ &\propto \int \frac{z^{2\left(\mathrm{i}\,u + \frac{l_1+l_2}{2} - 1\right)} C_1(y - v) C_2(x - v)}{(z - x)^{2\left(\mathrm{i}\,u + \frac{l_{21}}{2}\right)} (z - y)^{2\left(\mathrm{i}\,u + \frac{l_{12}}{2}\right)} (z - v)^{2\left(d - 1 + \frac{l_1+l_2}{2} - \mathrm{i}\,u\right)}} \frac{\mathrm{d}^d z \mathrm{d}^d v}{\pi^d} . \end{aligned}$$

[Derkachov, Ferrando, and Olivucci (2021)]

Orthogonality and Completeness

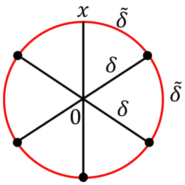
$$\langle \vec{u}_1, \ldots, \vec{u}_N; C | \vec{u}'_1, \ldots, \vec{u}'_N; C' \rangle = \frac{\prod_{k=1}^{N} \delta(\vec{u}_k - \vec{u}'_k)\ C \cdot C' + \ldots}{\mu(\vec{u}_1, \ldots, \vec{u}_N)},$$

where the measure is

$$\mu(\vec{u}_1, \ldots, \vec{u}_N) = \prod_{1 \leqslant j < k \leqslant N} \left[(u_j - u_k)^2 + \frac{(l_j - l_k)^2}{4} \right] \times \left[(u_j - u_k)^2 + \frac{(d - 2 + l_j + l_k)^2}{4} \right].$$

We conjecture that the eigenvectors we found generate the whole Hilbert space (completeness).

Applications: Periods of One-Wheel Graphs



$$= \langle x | \Lambda_1^M | x \rangle$$

$$= \frac{\Gamma\left(\frac{d-2}{2}\right)}{|x|^d} \sum_{l=0}^{+\infty} \left(l + \frac{d-2}{2} \right) \frac{(d-2)_l}{l!} \int_{-\infty}^{+\infty} \lambda_l^M(u) \frac{\mathrm{d}u}{2\pi}$$

For example, when $\delta = 1$, one has $\lambda_l^{-1}(u) \propto \left(u^2 + \frac{(2l+d-2)^2}{16} \right)$ and the above formula becomes

$$\frac{1}{|x|^4} \binom{2M-2}{M-1} \zeta_{2M-3} \quad \text{when} \quad d = 4\,,$$

[Belokurov and Ussyukina (1983)][Broadhurst (1985)]

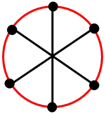
and

$$\frac{\pi^{\frac{1-M}{2}}}{|x|^3} \binom{2M-2}{M-1} (2^{2M-2} - 1) \zeta_{2M-2} \quad \text{when} \quad d = 3\,.$$

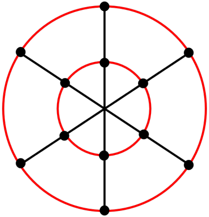
Applications: TBA for Wheel Graphs

In order to compute $\langle \mathrm{Tr}(Z^J)(x)\, \mathrm{Tr}(Z^{\dagger J})(y) \rangle$, one has to renormalise:

$1 + \xi^{2J}$



$+ \xi^{4J}$



$$+ \cdots = \sum_{N=0}^{+\infty} \xi^{2NJ} Tr\left(\Lambda_N^J\right) = Tr(e^{-J\widetilde{H}+JN\mu})$$

Interpretation: the N-th term of the sum is the contribution from the N-particle states to the mirror partition function in the thermodynamic limit with temperature $1/J$, chemical potential $\ln \xi^2$, and

$$\Lambda_N = e^{-\widetilde{H}}\Big|_{\text{N-particle sector}} .$$

$\Delta_{\mathrm{Tr}(Z^J)}$ = free energy density of mirror magnons at temperature $1/J$.
$\Rightarrow$ can be computed via the Thermodynamic Bethe Ansatz (TBA)

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The mirror scattering data are directly extracted from the eigenvectors.

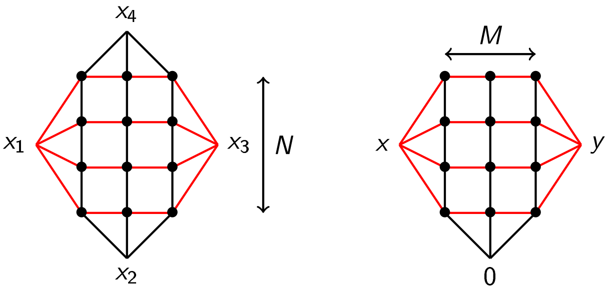
$$\langle x_1, x_2 | \vec{u}_1, \vec{u}_2; C_1, C_2 \rangle \underset{x_2^2 \gg x_1^2}{\propto} e^{2\,\mathrm{i}(u_1\sigma_1 + u_2\sigma_2)}\, C_1\left(\frac{x_1}{|x_1|}\right) C_2\left(\frac{x_2}{|x_2|}\right) + e^{2\,\mathrm{i}(u_2\sigma_1 + u_1\sigma_2)}\, [\mathrm{S}_{l_1,l_2}(u_1,u_2) C_1 \otimes C_2]\left(\frac{x_2}{|x_2|}, \frac{x_1}{|x_1|}\right),$$

where

$$\mathrm{S}_{l_1,l_2}(u_1,u_2) = \underbrace{\frac{f_{l_1}(u_1)}{f_{l_2}(u_2)} \mathcal{S}_{l_1,l_2}(u_1 - u_2)}_{\text{scalar phase}} \times \underbrace{\mathbb{R}_{l_1,l_2}(u_1 - u_2)}_{O(d,\mathbb{C})\text{-invariant R-matrix}}.$$

[Basso, Ferrando, Kazakov, and Zhong (2019)]

Applications: Basso–Dixon Integrals



[Basso and Dixon (2017)]

Right integral $= \langle x, \ldots, x| \left(\prod_{i=1}^{N} \hat{x}_{i-1,i}^{2\delta}\right) \Lambda_N^{M+1} |y, \ldots, y\rangle = I_{M,N}^{(d,\delta)}(x, y)$.

For $N = 2$ but arbitrary (d, δ), we were not able to simplify further than the following formula:

$$I_{M,2}^{(d,\delta)}(x,y) = \frac{\Gamma\left(\frac{d}{2}\right)^2}{2(x^2y^2)^{\tilde{\delta}}} \sum_{l_1,l_2=0}^{+\infty} \frac{(2l_1+d-2)(2l_2+d-2)}{(d-2)^2}$$

$$\times \sum_{m=0}^{\min(l_1,l_2)} a_{l_1,l_2,m} C_{l_1+l_2-2m}^{\left(\frac{d-2}{2}\right)}(\cos\theta) \int \left| \frac{\Gamma\left(\frac{l_1+l_2-2m+d-2}{2} + \mathrm{i}\, u_{12}\right)}{\Gamma\left(\frac{l_1+l_2-2m+2}{2} + \mathrm{i}\, u_{12}\right)} \right|^2$$

$$\times \left(\frac{x^2}{y^2}\right)^{\mathrm{i}(u_1+u_2)} [\lambda_{l_1}(u_1)\lambda_{l_2}(u_2)]^{M+2} \mu(\vec{u}_1, \vec{u}_2) \mathrm{d}u_1 \mathrm{d}u_2 \,,$$

where $\cos\theta = \frac{x\cdot y}{|x||y|}$, and $C_l^{(\alpha)}$ are the Gegenbauer polynomials.

[Derkachov, Ferrando, and Olivucci (2021)]

When $d = 4$, one has simply

$$I_{M,N}^{(4,\delta)}(x,y) = \frac{1}{N!(x^2y^2)^{\frac{N\tilde{\delta}}{2}}(e^{\mathrm{i}\,\theta} - e^{-\mathrm{i}\,\theta})^N}$$

$$\times \sum_{(a_1,\ldots,a_N)\in\mathbb{Z}^N} \int \mu(\vec{u}_1,\ldots,\vec{u}_N) \prod_{k=1}^{N} a_k\, \mathrm{e}^{\mathrm{i}\,a_k\theta} \left(\frac{x^2}{y^2}\right)^{\mathrm{i}\,u_k} \lambda_{a_k-1}^{M+N}(u_k)\mathrm{d}u_k\,.$$

[Basso and Dixon (2017)][Derkachov and Olivucci (2019)]

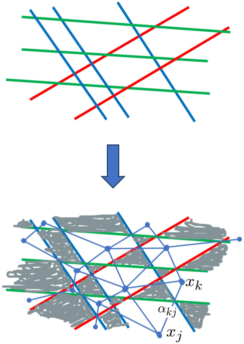
If we further impose $\delta = 1$ (isotropic lattice), then for $M \geqslant N$,

$$I_{M,N}^{(4,1)} = \det_{1\leqslant i,j\leqslant N} \left(c_{i,j}\, I_{M-N-1+i+j,1}^{(4,1)} \right)\,,$$

for some explicit $c_{i,j} \in \mathbb{Q}$. [Basso and Dixon (2017)]

[Basso, Dixon, Kosower, Krajenbrink, and Zhong (2021)]

Loom for CFTs



Generalization of the fishnet CFT based on arbitrary Baxter lattice (set of intersecting lines)

Same properties: non-unitary, conformal, integrable

[Kazakov and Olivucci (2022)]

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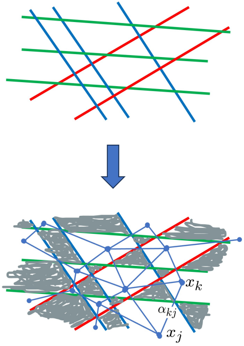

Feynman diagrams exhibit Yangian invariance

[Kazakov, Levkovich-Maslyuk, and Mishnyakov (2023)]

[Loebbert, Rüenaufer, and Stawinski (2025)]

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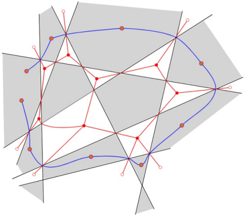
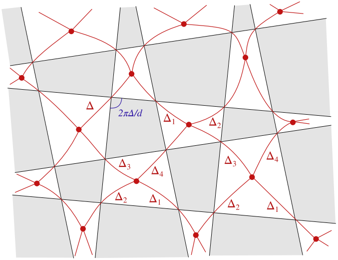


The Checkerboard CFT

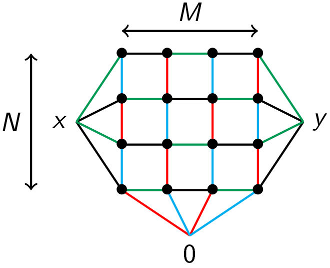
$$\mathcal{L}^{(CB)} = N_c \, \mathrm{Tr} \left[\sum_{j=1}^{4} \bar{Z}_j (-\partial_\mu \partial^\mu)^{w_j} Z_j - \xi_1^2 \, \bar{Z}_1 \bar{Z}_2 Z_3 Z_4 - \xi_2^2 \, Z_1 Z_2 \bar{Z}_3 \bar{Z}_4 \right] ,$$

with the constraint $w_1 + w_2 + w_3 + w_4 = d$. We will mostly use $\Delta_j = \frac{d}{2} - w_j$, which are the bare dimensions of the fields.

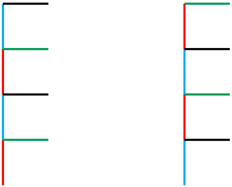


It encompasses the fishnet limit of the ABJM theory, and a theory with BFKL-like spectrum. [Alfimov, Ferrando, Kazakov, and Olivucci (2023)]

Checkerboard Analogues of Basso–Dixon Integrals



Main difference compared to the fishnet case: there are now 2 graph-building operators, Λ_N and Λ'_N, and they do not commute.



We do not diagonalise Λ_N and Λ'_N independently, but we construct 2 bases $\{|\vec{u}_1, \ldots, \vec{u}_N\rangle\}$ and $\{|\vec{u}_1, \ldots, \vec{u}_N\rangle'\}$ (let us focus on $d = 2$, with $\vec{u} = (u, m) \in \mathbb{R} \times \mathbb{Z}$) such that:

$$\Lambda_N\, |\vec{u}_1, \ldots, \vec{u}_N\rangle \propto |\vec{u}_1, \ldots, \vec{u}_N\rangle' \quad \text{and} \quad \Lambda'_N\, |\vec{u}_1, \ldots, \vec{u}_N\rangle' \propto |\vec{u}_1, \ldots, \vec{u}_N\rangle \ .$$

Symmetry, orthogonality, and completeness still hold.

$$I_{2L-1,N}(x,y) \propto \det_{1\leqslant i,j\leqslant N}\left(\sum_{m\in\mathbb{Z}}\int_{-\infty}^{+\infty}\left(\frac{m}{2}+\mathrm{i}\,u\right)^{i-1}\left(\frac{m}{2}-\mathrm{i}\,u\right)^{j-1} f_{N,2L}(\eta;\vec{u})\frac{\mathrm{d}u}{2\pi}\right),$$

where $(x,y)\in\mathbb{C}^2$, $\eta=\frac{x}{y}$,

$$f_{N,2L}(\eta;\vec{u}) = \eta^{\frac{m}{2}+\mathrm{i}\,u}\bar{\eta}^{-\frac{m}{2}+\mathrm{i}\,u}\lambda^{L+\lfloor\frac{N}{2}\rfloor}(\vec{u})\lambda'^{L+\lfloor\frac{N-1}{2}\rfloor}(\vec{u})\,,$$

and

$$\lambda(\vec{u}) = \frac{\Gamma\left(\frac{\Delta_1+m}{2}+\frac{\Delta_4-\Delta_2}{4}-\mathrm{i}\,u\right)\Gamma\left(\frac{\Delta_3+m}{2}+\frac{\Delta_4-\Delta_2}{4}+\mathrm{i}\,u\right)}{\Gamma\left(\frac{\Delta_2+m+1}{2}+\frac{\Delta_3-\Delta_1}{4}+\mathrm{i}\,u\right)\Gamma\left(\frac{\Delta_2+m+1}{2}+\frac{\Delta_1-\Delta_3}{4}-\mathrm{i}\,u\right)}\,.$$

The function λ' is obtained from λ through $(\Delta_1,\Delta_2)\leftrightarrow(\Delta_3,\Delta_4)$.

[Alfimov, Ferrando, Kazakov, and Olivucci (2023)]

One-Dimensional Integrals

Study of higher-point, track integrals using two techniques: SoV-like representations and differential equations (bootstrap)

[work in progress with F. Loebbert, A. Mierau, and S. Stawinski]

Study of higher-point, track integrals using two techniques: SoV-like representations and differential equations (bootstrap)

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Basic "SoV" relation:

$$\frac{1}{|x_{12}|^{2a}} = \sum_{\epsilon=0}^{1} \int \frac{\mathrm{sign}^{\epsilon}(x_{13}x_{23})}{|x_{13}|^{a+2\,\mathrm{i}\,u}|x_{23}|^{a-2\,\mathrm{i}\,u}} \frac{A_0(a)}{A_\epsilon(a/2-\mathrm{i}\,u)A_\epsilon(a/2+\mathrm{i}\,u)} \frac{\mathrm{d}u}{2\sqrt{\pi}},$$

where $A_\epsilon(x) = \Gamma((1+\epsilon)/2 - x)/\Gamma(\epsilon/2 + x)$.

Combining it with the star-triangle relation produces convenient representations of the Feynman integrals, which can be computed as series (sums over residues).

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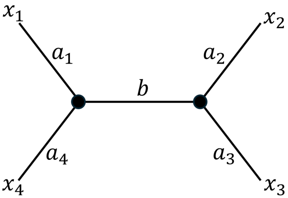
We computed all (non-conformal) integrals with 3,4,5,6 external points.

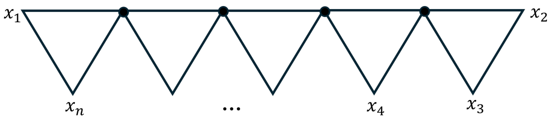
Example: H-integral



$$
\begin{aligned}
&= \frac{A_0(b)A_0(a_1)A_0(a_4)A_0(\tilde a_2+\tilde a_3-b)A_0(\tilde a_1+\tilde a_4)A_0(a_2+a_3)}{|x_{14}|^{2(a_1+a_4)-1}|x_{34}|^{2(b+a_2+a_3)-1}} F_2\left({}^{2(a_2+a_3-\tilde b);2\tilde a_4,2a_2}_{2(\tilde a_1+\tilde a_4),2(a_2+a_3)};\chi_1,\chi_2\right)\\
&+ \frac{A_0(a_1)A_0(a_2)A_0(a_3)A_0(a_4)A_0(\tilde a_1+\tilde a_4)A_0(\tilde a_2+\tilde a_3)}{|x_{14}|^{2(a_1+a_4)-1}|x_{34}|^{2b}|x_{23}|^{2(a_2+a_3)-1}} F_2\left({}^{2b;2\tilde a_4,2\tilde a_3}_{2(\tilde a_1+\tilde a_4),2(\tilde a_2+\tilde a_3)};\chi_1,\chi_2\right)\\
&+ \frac{A_0(b)A_0(3/2-b-\sum_i a_i)A_0(a_1+a_4)A_0(a_2+a_3)}{|x_{34}|^{2(b+\sum_i a_i-1)}} F_2\left({}^{2(\sum_i a_i+b-1);2a_1,2a_2}_{2(a_1+a_4),2(a_2+a_3)};\chi_1,\chi_2\right)\\
&+ \frac{A_0(b)A_0(a_2)A_0(a_3)A_0(\tilde a_1+\tilde a_4-b)A_0(a_1+a_4)A_0(\tilde a_2+\tilde a_3)}{|x_{34}|^{2(b+a_1+a_4)-1}|x_{23}|^{2(a_2+a_3)-1}} F_2\left({}^{2(a_1+a_4-\tilde b);2a_1,2\tilde a_3}_{2(a_1+a_4),2(\tilde a_2+\tilde a_3)};\chi_1,\chi_2\right),
\end{aligned}
$$

where $\chi_1 = x_{14}/x_{34}$, $\chi_2 = x_{23}/x_{43}$, and $\tilde a = 1/2 - a$.

Example: Conformal Partial Waves in the Comb Channel



$$= V_n \prod_{i=1}^{n-2} A_0\left(\frac{1+g_{i-1,i}-h_{n+1-i}}{2}\right) \prod_{j=1}^{n-3} \sum_{\delta_j \in \{g_{n-2-j},1-g_{n-2-j}\}} A_0(\delta_j)|\chi_j|^{\delta_j}$$

$$\times \prod_{j=2}^{n-3} A_0\left(\frac{h_{j+2}+1-\delta_{j-1}-\delta_j}{2}\right) A_0\left(\frac{h_{32}+1-\delta_1}{2}\right) A_0\left(\frac{h_{n1}+1-\delta_{n-3}}{2}\right)$$

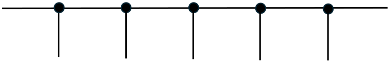
$$\times F_K\left({}^{\delta_1+h_{23};\delta_1+\delta_2-h_3,\dots,\delta_{n-4}+\delta_{n-3}-h_{n-2};\delta_{n-3}+h_{1n}}_{2\delta_1,\dots,2\delta_{n-3}};\chi_1,\dots,\chi_{n-3}\right)\,,$$

where V_n is a kinematical prefactor, $\chi_j = x_{j+1,j+2}\,x_{j+3,j+4}/x_{j+1,j+3}\,x_{j+2,j+4}$, and

$$F_K\left({}^{a_1;b_1,\dots,b_{n-1};a_2}_{c_1,\dots,c_n};x_1,\dots,x_n\right) = \sum_{m_1,\dots,m_n=0}^{+\infty} \frac{(a_1)_{m_1}\prod_{i=1}^{n-1}(b_i)_{m_i+m_{i+1}}(a_2)_{m_n}}{\prod_{i=1}^{n}(c_i)_n} \prod_{i=1}^{n} \frac{x_i^{m_i}}{m_i!}\,.$$

[Rosenhaus (2018)]

Triangle-track integrals involve more complicated series



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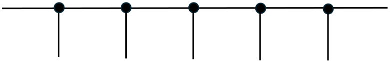
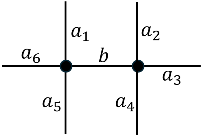


Generally, several SoV-like representations can be derived: how can we find the simplest one?
For instance, for conformal integrals:



$$a_1 + a_5 + a_6 + b = 1$$
$$a_2 + a_3 + a_4 + b = 1$$

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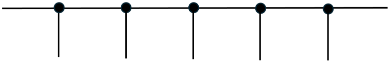

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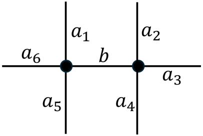
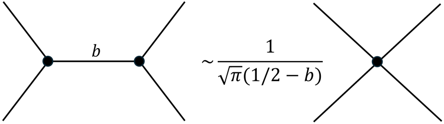


$$a_1 + a_5 + a_6 + b = 1$$
$$a_2 + a_3 + a_4 + b = 1$$

Other degeneracies:



Future Directions

- Corrections (in g) to the fishnet limit can also be studied using $\widehat{\mathcal{H}}$
- What about structure constants involving operators with different double-scaling limits?
- Extra simplification in $d \notin \{2, 4\}$? Basso–Dixon integrals in 1D?
- Hexagonalisation and application to other classes of fishnet integrals
  [Basso, Komatsu, and Vieira (2015)][Basso, Caetano, and Fleury (2018)]
  [Aprile and Olivucci (2023)]

Thank you for your attention!