Exact results in superfishnet theory and relaxation of the double-scaling limit

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MK: 2410.18176 [hep-th]
MK,Staudacher: 2408.05805 [hep-th]
PhD thesis: soon on the arXiv

Fishnet QFTs: Integrability, periods and beyond
University of Southampton
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INFN PADOVA
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Sezione di Padova

Introduction

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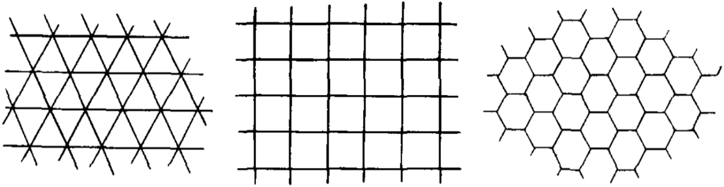

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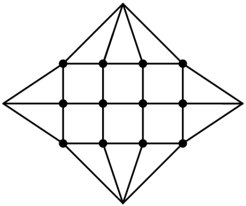

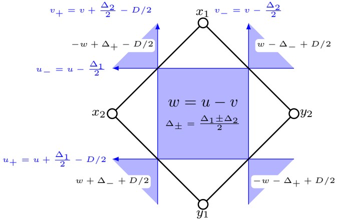

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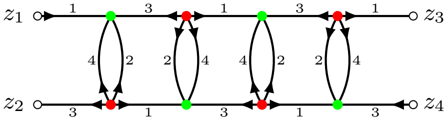
- Many **exact computations are possible by integrability** in analogy to spin-chains and lattice models
- In order to **learn about the mother theories**, the decoupled degrees of freedom have to be restored, which seemingly makes the Feynman diagrams more complicated
- We propose to work with **supergraphs as an ordering principle**. It allows us to turn to less radical deformations while maintaining the regular structure in the diagrams of perturbative expansions such that methods developed in the fishnet models can be lifted "closer" to $\mathcal{N} = 4$ SYM and ABJM. Even gauge fields can be restored in a controlled way.

Overview

Plan of the talk:

1. Derivation of the superspace action of superfishnet (and super brick wall model)
2. Super-Feynman rules and superspace integral relations
3. Critical coupling of superfishnets and super brick walls
4. Calculation of all-loop scaling dimensions of various operators and OPE coefficient in both theories



5. Relaxing the double-scaling limit and perturbative reappearance of gauge fields

β-deforming ABJM theory

ABJM in $\mathcal{N} = 2$ superspace: [Benna,Klebanov,Klose,Smedback'08]

$$S_{\mathrm{ABJM}} = -\mathrm{i}\,\frac{k}{\lambda} \cdot S_{\mathrm{CS}}\left[\mathcal{V}, \hat{\mathcal{V}}\right] + \mathrm{N} \cdot S_{\mathrm{mat}}\left[\mathcal{Z}, \mathcal{W}, \mathcal{V}, \hat{\mathcal{V}}\right] + \mathrm{N}\lambda^2 \cdot S_{\mathrm{pot}}\left[\mathcal{Z}, \mathcal{W}\right]$$

$$S_{\mathrm{mat}} = \int \mathrm{d}^3x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ \mathrm{tr}\left[-\bar{\mathcal{Z}}_A \mathrm{e}^{-\lambda\mathcal{V}} \mathcal{Z}^A \mathrm{e}^{\lambda\hat{\mathcal{V}}} - \bar{\mathcal{W}}^A \mathrm{e}^{-\lambda\hat{\mathcal{V}}} \mathcal{W}_A \mathrm{e}^{\lambda\mathcal{V}}\right]$$

$$S_{\mathrm{pot}} = \int \mathrm{d}^3x\, \mathrm{d}^2\theta\ \frac{1}{4}\,\varepsilon_{AC}\varepsilon^{BD}\ \mathrm{tr}\left[\mathcal{Z}^A \mathcal{W}_B \mathcal{Z}^C \mathcal{W}_D\right] + \mathrm{h.c.}$$

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$$S_{\rm mat} = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\;\mathrm{tr}\left[-\bar{\mathcal{Z}}_A \mathrm{e}^{-\lambda\mathcal{V}}\mathcal{Z}^A\mathrm{e}^{\lambda\hat{\mathcal{V}}} - \bar{\mathcal{W}}^A\mathrm{e}^{-\lambda\hat{\mathcal{V}}}\mathcal{W}_A\mathrm{e}^{\lambda\mathcal{V}}\right]$$

$$S_{\rm pot} = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\;\frac{1}{4}\,\varepsilon_{AC}\varepsilon^{BD}\;\mathrm{tr}\left[\mathcal{Z}^A\mathcal{W}_B\mathcal{Z}^C\mathcal{W}_D\right] + \mathrm{h.c.}$$

β-deformation: [Imeroni'08]

$$\Phi_1\cdots\Phi_p \to \mathrm{e}^{-\frac{\mathrm{i}}{2}\sum_{m>n}^{p}\beta\cdot\varepsilon_{ij}\,q^i_{\Phi_m}q^j_{\Phi_n}}\;\Phi_1\cdots\Phi_p$$

	$\mathcal{Z}^1$	$\mathcal{Z}^2$	$\mathcal{W}_1$	$\mathcal{W}_2$	$\bar{\mathcal{Z}}_1$	$\bar{\mathcal{Z}}_2$	$\bar{\mathcal{W}}^1$	$\bar{\mathcal{W}}^2$	$\mathcal{V}$	$\hat{\mathcal{V}}$
q^1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
q^2	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$R=q^3$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0

Double-scaling the β-deformation of ABJM theory

β-deformed ABJM:

$$S_{\mathrm{ABJM},\beta} = -\mathrm{i}\,\frac{k}{\lambda}\cdot S_{\mathrm{CS}}\left[\mathcal{V},\hat{\mathcal{V}}\right] + \mathrm{N}\cdot S_{\mathrm{mat}}\left[\mathcal{Z},\mathcal{W},\lambda\mathcal{V},\lambda\hat{\mathcal{V}}\right]$$
$$+\,\mathrm{N}\lambda^2\int \mathrm{d}^3x\,\mathrm{d}^2\theta\;\tfrac{1}{2}\,\mathrm{tr}\left[q\cdot\mathcal{Z}^1\mathcal{W}_2\mathcal{Z}^2\mathcal{W}_1 - q^{-1}\cdot\mathcal{Z}^1\mathcal{W}_1\mathcal{Z}^2\mathcal{W}_2\right] + \mathrm{h.c.}$$

with $q := \mathrm{e}^{\frac{\mathrm{i}}{4}\beta}$.

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Double-scaling limit: $\lambda \to 0$ and $q \to \infty$ while $\xi := \frac{1}{2}q\lambda^2$ is kept finite

- Matter-part of the action becomes independent of $\mathcal{V}$, $\hat{\mathcal{V}}$, hence S_{CS} decouples.
- The superpotential becomes non-unitary

$$\mathrm{N}\xi\int \mathrm{d}^3x\,\mathrm{d}^2\theta\,\mathrm{tr}\left[\mathcal{Z}^1\mathcal{W}_2\mathcal{Z}^2\mathcal{W}_1\right] + \mathrm{N}\xi\int \mathrm{d}^3x\,\mathrm{d}^2\bar{\theta}\,\mathrm{tr}\left[\bar{\mathcal{Z}}_1\bar{\mathcal{W}}^2\bar{\mathcal{Z}}_2\bar{\mathcal{W}}^1\right]$$

Superfishnet and super brick wall theory

After renaming the fields, the **superfishnet theory** reads
[MK'24;Caetano,Gürdoğan,Kazakov'18]

$$S_{\mathrm{SFN}} = \mathrm{N} \int \mathrm{d}^3 x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta}$$

$$\cdot \, \mathrm{tr} \left[- \sum_{i=1}^{4} \Phi_i^\dagger \Phi_i + \xi \cdot \bar{\theta}^2 \, \Phi_1 \Phi_2 \Phi_3 \Phi_4 + \xi \cdot \theta^2 \, \Phi_1^\dagger \Phi_2^\dagger \Phi_3^\dagger \Phi_4^\dagger \right]$$

By the analogous procedure obtain **super brick wall theory** from $\mathcal{N} = 4$ SYM [MK,Staudacher'24]

$$S_{\mathrm{SBW}} = \mathrm{N} \int \mathrm{d}^4 x \, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}$$

$$\cdot \, \mathrm{tr} \left[\sum_{i=1}^{3} \Phi_i^\dagger \Phi_i + \mathrm{i}\xi \cdot \bar{\theta}^2 \, \Phi_1 \Phi_2 \Phi_3 + \mathrm{i}\xi \cdot \theta^2 \, \Phi_1^\dagger \Phi_2^\dagger \Phi_3^\dagger \right]$$

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$$S_{\mathrm{SFN}} = \mathrm{N} \int \mathrm{d}^3x \ \mathrm{tr} \Big\{ Y_M^\dagger \Box Y^M + \mathrm{i}\, \Psi^{\dagger M\alpha} \gamma^\mu_{\alpha\beta} \partial_\mu \Psi_M^\beta$$
$$+ \xi^2 \left[Y^1 Y_4^\dagger Y^2 Y_1^\dagger Y^4 Y_2^\dagger + Y^1 Y_4^\dagger Y^3 Y_1^\dagger Y^4 Y_3^\dagger + Y^2 Y_3^\dagger Y^1 Y_2^\dagger Y^3 Y_1^\dagger + Y^2 Y_3^\dagger Y^4 Y_2^\dagger Y^3 Y_4^\dagger \right]$$
$$- \mathrm{i}\xi \left[\Psi_2 \Psi^{\dagger 3} Y^2 Y_3^\dagger - \Psi^{\dagger 3} \Psi_1 Y_3^\dagger Y^1 + \Psi_1 \Psi^{\dagger 4} Y^1 Y_4^\dagger - \Psi_4 \Psi^{\dagger 2} Y^4 Y_2^\dagger \right]$$
$$+ \mathrm{i}\xi \left[\Psi^{\dagger 2} \Psi_3 Y_2^\dagger Y^3 - \Psi_3 \Psi^{\dagger 1} Y^3 Y_1^\dagger + \Psi^{\dagger 1} \Psi_4 Y_1^\dagger Y^4 - \Psi^{\dagger 4} \Psi_2 Y_4^\dagger Y^2 \right]$$
$$+ \mathrm{i}\xi \left[\Psi_2 Y_4^\dagger \Psi_1 Y_3^\dagger + \Psi^{\dagger 3} Y^2 \Psi^{\dagger 4} Y^1 - \Psi^{\dagger 2} Y^4 \Psi^{\dagger 1} Y^3 - \Psi_3 Y_2^\dagger \Psi_4 Y_1^\dagger \right] \Big\}$$

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$$S_{\mathrm{SBW}} = \mathrm{N} \int \mathrm{d}^4x \ \mathrm{tr} \Big\{ \sum_{i=1}^{3} \left[\phi_i^\dagger \Box \phi_i - \mathrm{i}\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i \right] + \xi^2 \left[\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger + \phi_3\phi_1\phi_3^\dagger\phi_1^\dagger + \phi_2\phi_3\phi_2^\dagger\phi_3^\dagger \right]$$
$$- \mathrm{i}\xi \left[\phi_1\psi_2\psi_3 + \phi_2\psi_3\psi_1 + \phi_3\psi_1\psi_2 \right] - \mathrm{i}\xi \left[\phi_1^\dagger \bar{\psi}_2\bar{\psi}_3 + \phi_2^\dagger \bar{\psi}_3\bar{\psi}_1 + \phi_3^\dagger \bar{\psi}_1\bar{\psi}_2 \right] + \mathcal{L}_{\mathrm{dt}} \Big\}$$

Superfield propagator

Chiral superfield at point $z = (x, \theta, \bar{\theta})$ in superspace

$$\Phi_i(z) = \mathrm{e}^{\mathrm{i}\theta\gamma^\mu\bar{\theta}\partial_\mu}\left[\phi_i(x) + \sqrt{2}\,\theta\psi_i(x) + \theta^2 F_i(x)\right]$$

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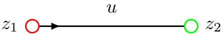
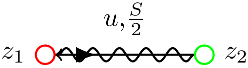
Generalized superfield propagator from bosonic and fermionic propagators in three- and four dimensions, respectively:

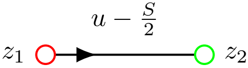
$$\left\langle \Phi_i(z_1)\Phi_j^\dagger(z_2)\right\rangle_u = \mathrm{e}^{\mathrm{i}\left[\theta_1\gamma^\mu\bar{\theta}_1+\theta_2\gamma^\mu\bar{\theta}_2-2\theta_1\gamma^\mu\bar{\theta}_2\right]\partial_{1,\mu}}\frac{\delta_{ij}}{\left[x_{12}^2\right]^u} = \frac{\delta_{ij}}{\left[x_{1\bar{2}}^2\right]^u}$$

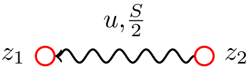
$=$ 

Spinning propagator and auxiliary 2pt-function



$$:= \frac{\Gamma(u-\frac{S}{2})}{\Gamma(u+\frac{S}{2})}\frac{\partial_1^{\mu_1}\cdots\partial_1^{\mu_S}}{(-2)^S}$$





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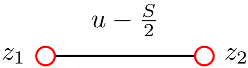

Super-Feynman rules

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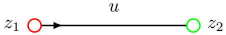
<tr><td colspan="2">
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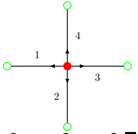
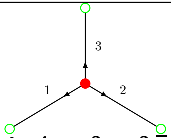
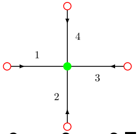
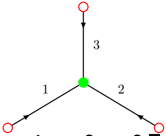
<tr><td>$u = \frac{1}{2}$</td><td>$u = 1$</td></tr>
<tr><td>

$\sim \mathrm{i}\xi \int \mathrm{d}^3 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ \delta^{(2)}(\bar{\theta})$</td><td>

$\sim -\xi \int \mathrm{d}^4 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ \delta^{(2)}(\bar{\theta})$</td></tr>
<tr><td>

$\sim \mathrm{i}\xi \int \mathrm{d}^3 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ \delta^{(2)}(\theta)$</td><td>

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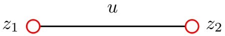
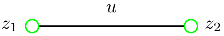
<tr><td colspan="2">$\frac{\theta_{12}^2}{\left[x_{12}^2\right]^u} =$ 

, $\frac{\bar{\theta}_{12}^2}{\left[x_{12}^2\right]^u} =$ 
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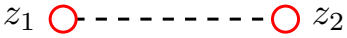
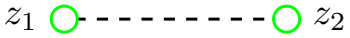
<tr><td colspan="2">$\delta\left(z_{12}\right) =$ 

, $\delta\left(\bar{z}_{12}\right) =$ 
</td></tr>
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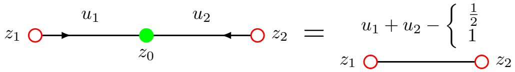
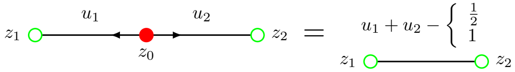
Useful superintegral relations

Supersymmetric **chain relations** for $\left\{ \begin{array}{l} \text{superfishnet} \\ \text{super brick wall} \end{array} \right\}$ theory:



$$\cdot \left\{ \begin{array}{l} 4\mathrm{i}\, r_0(2 - u_1 - u_2, u_1, u_2) \\ -4\, r_0(3 - u_1 - u_2, u_1, u_2) \end{array} \right.$$



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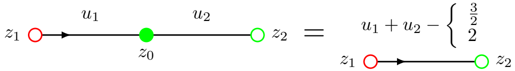
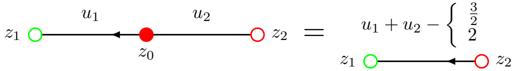


$$\cdot \left\{ \begin{array}{l} \mathrm{i}\, r_0(3 - u_1 - u_2, u_1, u_2) \\ r_0(4 - u_1 - u_2, u_1, u_2) \end{array} \right.$$

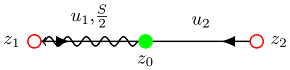


$$\cdot \left\{ \begin{array}{l} \mathrm{i}\, r_0(3 - u_1 - u_2, u_1, u_2) \\ r_0(4 - u_1 - u_2, u_1, u_2) \end{array} \right.$$

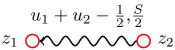
with the abbreviations $r_\ell(u_1, u_2, u_3) := \pi^2 a_0(u_1) a_\ell(u_2) a_\ell(u_3)$,
$a_\ell(u) := \frac{\Gamma(\frac{D}{2} - u + \ell)}{\Gamma(u + \ell)}$

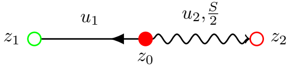
Useful superintegral relations

Spinning chain relations

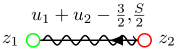


$$= 4\, r_{\frac{S}{2}}(u_2, u_1, 2 - u_1 - u_2)$$



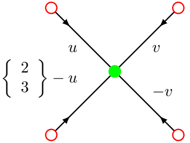


$$= r_{\frac{S}{2}}(u_1, u_2, 3 - u_1 - u_2)$$

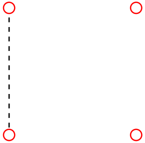


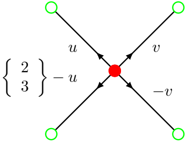
Super x-unity

[MK'24;MK,Staudacher'24]

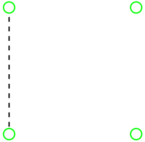


$$= \left\{ \begin{matrix} 4\,\pi^3 a_0(u)\, a_0(2-u) \\ -4\,\pi^4 a_0(u)\, a_0(3-u) \end{matrix} \right\} \cdot$$





$$= \left\{ \begin{matrix} 4\,\pi^3 a_0(u)\, a_0(2-u) \\ -4\,\pi^4 a_0(u)\, a_0(3-u) \end{matrix} \right\} \cdot$$



A star-3×triangle relation?

Osborn's formula (originally in $D = 4$ $\mathcal{N} = 1$)
[Osborn:9808041][Dolan,Osborn:0006098] adapted to $D = 3$ $\mathcal{N} = 2$:

$$\int \mathrm{d}^3 x_0 \, \mathrm{d}^2\theta_0 \, \mathrm{d}^2\bar{\theta}_0 \; \delta^{(2)}(\bar{\theta}_0) \, \frac{1}{\left[x^2_{0\bar{1}}\right]^{u_1}} \frac{1}{\left[x^2_{0\bar{2}}\right]^{u_2}} \frac{1}{\left[x^2_{0\bar{3}}\right]^{u_3}}$$

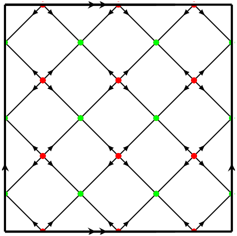
$$\stackrel{u_1+u_2+u_3=2}{=} 8\, r_0(u_1, u_2, u_3) \, \frac{\bar{\theta}_{13}\bar{\theta}_{23}x^2_{12,-} + \bar{\theta}_{12}\bar{\theta}_{32}x^2_{13,-} + \bar{\theta}_{12}\bar{\theta}_{13}x^2_{23,-}}{[x^2_{12,-}]^{3/2-u_3}[x^2_{13,-}]^{3/2-u_2}[x^2_{23,-}]^{3/2-u_1}}$$

with $x^{\mu}_{ij,+} := x^{\mu}_{i,+} - x^{\mu}_{j,+}$, $x^{\mu}_{\pm} := x^{\mu} \pm \mathrm{i}\theta\gamma^{\mu}\bar{\theta}$, $\bar{\theta}_{ij} := \bar{\theta}_i - \bar{\theta}_j$

Vacuum superdiagrams

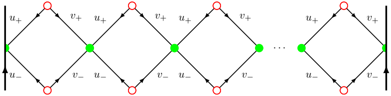
Row-matrices are stacked up and periodically identified to form **toroidal vacuum superdiagrams** of the superfishnet theory:

$$Z_{3,3}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \mathrm{Tr}\left[T_3 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}^3 \right] =$$



with generalized row-matrix

$$T_N \begin{pmatrix} u_+ & v_+ \\ u_- & v_- \end{pmatrix} =$$



Critical coupling

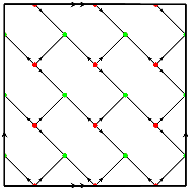
The **critical coupling = (radius of convergence)$^{-1}$** is the vacuum graphs in the "thermodynamic limit", obtained by the **method of inversion relations** [Zamolodchikov'80;MK,Staudacher'23]

$$\xi_{\mathrm{cr}}^{\mathrm{SFN}} = \lim_{M,N\to\infty} \left| Z_{MN} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right|^{\frac{1}{2MN}} = \frac{1}{\pi^3 \sqrt{2}\, |\eta(\mathrm{i})|^2}$$

[MK'24]

Similarly, for the super brick wall theory:

$$Z_{3,3} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix} \right) =$$



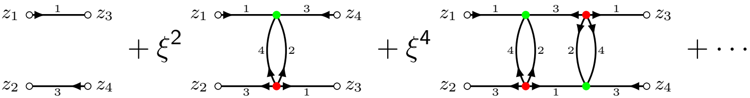
Critical coupling: $\xi_{\mathrm{cr}}^{\mathrm{SBW}} = \frac{3^{9/8}}{4\pi^3\, |\eta(\mathrm{e}^{\mathrm{i}\pi/3})|}$

[MK,Staudacher'24]

All-loop anomalous dimensions in the superfishnet theory

Extract **anomalous dimension** of $\mathrm{tr}\left[\Phi_1^\dagger \partial^S \Phi_3\right]$ from 4pt.-function, similar to [Gromov,Kazakov,Korchemsky'18]

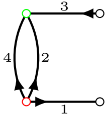
$$\left\langle \mathrm{tr}\left[\Phi_1(z_1)\Phi_3^\dagger(z_2)\right] \mathrm{tr}\left[\Phi_1^\dagger(z_3)\Phi_3(z_4)\right]\right\rangle =$$

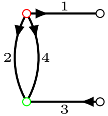


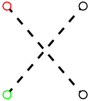
$$+\ \xi^2 \quad +\ \xi^4 \quad + \cdots$$

$$= x_{12}^2 \bar{\mathbb{H}} \frac{1 + \xi^2 \mathbb{H} \circ \mathbb{P}}{1 - \xi^4 \mathbb{H} \circ \bar{\mathbb{H}}}$$

with graph-building operators

$\mathbb{H} =$ 

, $\bar{\mathbb{H}} =$ 

, $\mathbb{P} =$ 

All-loop anomalous dimensions in the superfishnet theory

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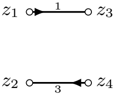
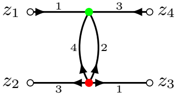
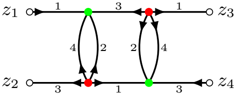


$+\,\xi^2$



$+\,\xi^4$



$+\cdots$

$$= x_{1\bar{2}}^2 \bar{\mathbb{H}} \frac{1+\xi^2 \mathbb{H}\circ\mathbb{P}}{1-\xi^4 \mathbb{H}\circ\bar{\mathbb{H}}} \sim \sum\!\!\!\!\!\!\!\int_{\Delta,S} \; |\bar{\Omega}_{\Delta,S,0}\rangle\langle\Omega_{\Delta,S,0}| \frac{E_0(\Delta,S)}{1-\xi^2 E_0(\Delta,S)}$$

with a complete set of eigenfunctions $\mathbb{1} = \sum\!\!\!\!\!\!\!\int_{\Delta,S} \; |\bar{\Omega}_{\Delta,S,R}\rangle\langle\Omega_{\Delta,S,R}|$
and eigenvalue $\langle\Omega_{\Delta,S,0}|\mathbb{H}\circ\mathbb{P} = \langle\Omega_{\Delta,S,0}|E_0(\Delta,S)$.
The **pole condition**

$$1 = \xi^2 E_0(\Delta,S)$$

signals the exchange of the operator $\mathrm{tr}\left[\Phi_1^\dagger(z_3)\partial^S\Phi_3(z_4)\right]$

Zero-magnon case

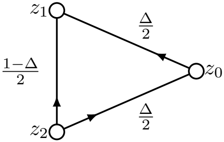
The $D = 3$ $\mathcal{N} = 2$ superconformal eigenfunctions with $R = 0$ are [Chang,Colin-Ellerin,Peng,Rangamani'21]

$$\Omega_{\Delta,S,0}(z_1,z_2;z_0) = \frac{C_{\Phi_1^\dagger \Phi_3 \mathcal{O}}}{\left[x^2_{2\bar{1}}\right]^{\Delta_\Phi - \frac{\Delta}{2} + \frac{S}{2}} \left[x^2_{0\bar{1}}\right]^{\frac{\Delta}{2} + \frac{S}{2}} \left[x^2_{2\bar{0}}\right]^{\frac{\Delta}{2} + \frac{S}{2}}} \left[X_{3,-}^{\mu_1} \cdots X_{3,-}^{\mu_S} - \mathrm{traces}\right]$$

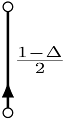
Zero-magnon case

The $D = 3$ $\mathcal{N} = 2$ superconformal eigenfunctions with $R, S = 0$

$\Omega_{\Delta,0,0} =$



$\underset{\theta_0,\bar{\theta}_0=0}{\overset{x_0\to\infty}{\sim}} \Psi_{\frac{1-\Delta}{2}} :=$



Zero-magnon case

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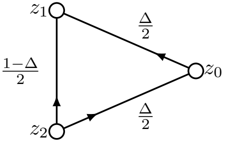
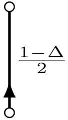
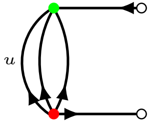
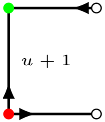
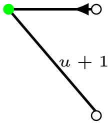
$\Omega_{\Delta,0,0} =$ 

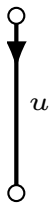
$\overset{x_0 \to \infty,\ \theta_0, \bar{\theta}_0 = 0}{\sim}$ $\Psi_{\frac{1-\Delta}{2}} :=$ 

Calculate the eigenvalue of $\mathbb{H} \circ \mathbb{P}$ and $\mathbb{H} \circ \bar{\mathbb{H}}$ from

$\langle \Psi_u | \mathbb{H} =$ 

$=$ 

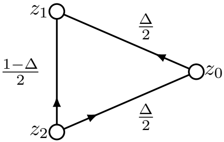
$\sim$ 

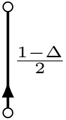
$\sim$ 

$\sim \mathrm{E}_0(u) \langle \Psi_u^\dagger |$

Zero-magnon case

The $D = 3$ $\mathcal{N} = 2$ superconformal eigenfunctions with $R, S = 0$

$\Omega_{\Delta,0,0} =$ 

$\underset{\theta_0,\bar{\theta}_0=0}{\overset{x_0\to\infty}{\sim}}$ $\Psi_{\frac{1-\Delta}{2}} :=$ 

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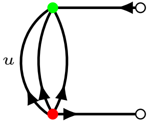
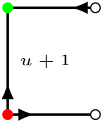
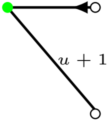
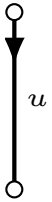
$\langle \Psi_u | \mathbb{H} =$ 

$=$ 

$\sim$ 

$\sim$ 

$\sim \mathrm{E}_0(u) \langle \Psi_u^\dagger |$

The eigenvalue for generic S is

$$E_0(\Delta, S) = \frac{4\pi^4}{(1 + S - \Delta)(S + \Delta)}$$

with $E_0(1 - \Delta, S) = E_0(\Delta, S)$

Zero-magnon case

Inverting the pole condition gives the **exact scaling dimensions** [MK'24,MK:PhD thesis]

$$\Delta = 1 + \frac{1}{2}\left(-1 + 2\sqrt{(S+\tfrac{1}{2})^2 - 4\pi^4\xi^2}\right)$$

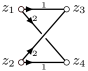
in agreement with [Caetano,Gürdoğan,Kazakov'16] for $S = 0$. The eigenfunctions $|\bar{\Omega}_{\Delta,S,R}\rangle\langle\Omega_{\Delta,S,R}|$ are expressed in terms of the superconformal blocks [Bobev,El-Showk,Mazac,Paulos'15] and by the residue theorem the integral over Δ is performed to find the **exact OPE coefficient** [MK:PhD thesis]

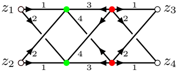
$$C_{\Delta,S} = -2^{S-1-2\Delta}\pi\frac{\Gamma(S+\frac{3}{2})\Gamma(\Delta)\Gamma(\frac{S-\Delta+2}{2})\Gamma(\frac{S+\Delta}{2})}{\Gamma(S+1)\Gamma(\Delta+\frac{1}{2})\Gamma(\frac{S-\Delta+3}{2})\Gamma(\frac{S+\Delta+1}{2})}$$

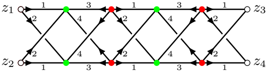
Two-magnon case

Extract anomalous dimension of $\mathrm{tr}\left[\Phi_1\Phi_2\Phi_1\Phi_2\right]$ from

$$\left\langle \mathrm{tr}\left[\Phi_2(z_1)\Phi_1(z_1)\Phi_2(z_2)\Phi_1(z_2)\right] \mathrm{tr}\left[\Phi_1^\dagger(z_3)\Phi_2^\dagger(z_3)\Phi_1^\dagger(z_4)\Phi_2^\dagger(z_4)\right]\right\rangle =$$



$+ \xi^4$


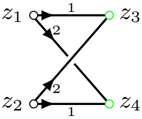
$+ \xi^8$


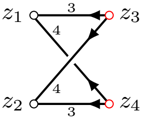
$+ \cdots$

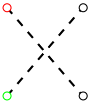
$$+ (z_3 \leftrightarrow z_4)$$

$$= \frac{\mathbb{H} \circ (1+\mathbb{P})}{1 - \xi^4 \mathbb{H} \circ \bar{\mathbb{H}}}$$

with graph-building operators

$\mathbb{H} =$


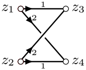
, $\bar{\mathbb{H}} =$


, $\mathbb{P} =$


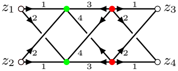
Two-magnon case

Extract anomalous dimension of $\mathrm{tr}\,[\Phi_1\Phi_2\Phi_1\Phi_2]$ from

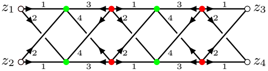
$$\left\langle \mathrm{tr}\,[\Phi_2(z_1)\Phi_1(z_1)\Phi_2(z_2)\Phi_1(z_2)]\,\mathrm{tr}\left[\Phi_1^\dagger(z_3)\Phi_2^\dagger(z_3)\Phi_1^\dagger(z_4)\Phi_2^\dagger(z_4)\right]\right\rangle =$$



$+\,\xi^4$



$+\,\xi^8$



$+\cdots$

$$+\,(z_3 \leftrightarrow z_4)$$

$$= \frac{\mathbb{H}\circ(1+\mathbb{P})}{1-\xi^4\mathbb{H}\circ\bar{\mathbb{H}}} = \sum\!\!\!\!\!\!\!\int_{\Delta,S} |\bar{\Omega}_{\Delta,S,-2}\rangle\langle\Omega_{\Delta,S,-2}|\frac{E_2(\Delta,S)}{1-\xi^4E_2(\Delta,S)^2}$$

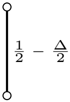
with a complete set of eigenfunctions $\mathbb{1} = \sum\!\!\!\!\!\!\!\int_{\Delta,S} |\bar{\Omega}_{\Delta,S,R}\rangle\langle\Omega_{\Delta,S,R}|$ and eigenvalue $\langle\Omega_{\Delta,S,-2}|\mathbb{H}\circ\bar{\mathbb{H}} = \langle\Omega_{\Delta,S,-2}|E_2(\Delta,S)^2$. The spinless **pole condition** is

$$1 = \xi^4 E_2(\Delta,0)^2$$

Two-magnon case

I am unaware of the exact form of the $R = -2$ eigenfunction, however, it can be obtained in the limit

$$\Omega_{\Delta,0,-2} \overset{\substack{x_0 \to \infty \\ \theta_0, \bar{\theta}_0 = 0}}{\sim} \Psi_{\frac{1}{2} - \frac{\Delta}{2}} :=$$



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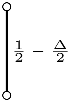
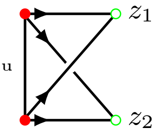


Calculate the eigenvalue of $\mathbb{H} \circ \bar{\mathbb{H}}$ from

$$\langle \Psi_u | \mathbb{H} =$$



$$\sim \bar{\theta}_{12}^2 \Box_1 \, \mathrm{kite}^{(3)}(x_{12}^2, u) \sim \mathrm{E}_2(u) \langle \Psi_u^\dagger |$$

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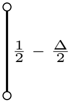

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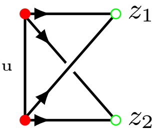
$$\langle\Psi_u|\mathbb{H} =$$



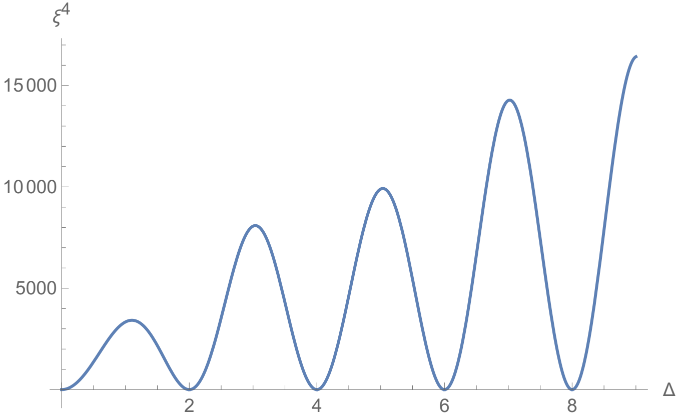
$$\sim \bar\theta_{12}^2\Box_1\,\mathrm{kite}^{(3)}(x_{12}^2,u) \sim \mathrm{E}_2(u)\langle\Psi_u^\dagger|$$

One finds the **eigenvalue** [MK'24]

$$E_2(\Delta) = \frac{\csc(\pi(\frac{\Delta}{2}+1))\,\Gamma(\frac{\Delta}{2}+1)}{32\sqrt{\pi}\,\Gamma(\frac{\Delta}{2}+\frac{3}{2})} - \frac{{}_3F_2\left(1,1,\frac{\Delta}{2}+\frac{3}{2};\frac{\Delta}{2}+2,\frac{\Delta}{2}+\frac{5}{2};1\right)}{16\pi^2\,(\Delta+3)\left(\frac{\Delta}{2}+1\right)}$$

Two-magnon case

Plot of $\xi^4 = \frac{1}{E_2(\Delta)^2}$:



Zeros correspond to classical scaling dimensions of $\Box^n \mathrm{tr}\left[\Phi_1\Phi_2\Phi_1\Phi_2\right]$

Two-magnon case

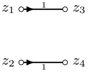
The small-coupling expansion of the first three operator's scaling dimension:

$$
\begin{aligned}
\Delta^{(2)} &= 2 \pm \frac{\xi^2}{12} - \frac{\xi^4}{576} \pm \frac{18\pi^2 - 97}{248832}\xi^6 \\
&\quad + \frac{3803 - 2268\zeta_3 + 18\pi^2(\log(4096) - 19)}{35831808}\xi^8 + \mathcal{O}(\xi^{10}) \ , \\
\Delta^{(4)} &= 4 \pm \frac{4\xi^2}{15} - \frac{\xi^4}{7200} \pm \frac{28800\pi^2 - 191191}{777600000}\xi^6 \\
&\quad \frac{191678057 - 145152000\zeta_3 + 28800\pi^2(480\log(2) - 421)}{5598720000000}\xi^8 + \mathcal{O}(\xi^{10}) \ , \\
\Delta^{(6)} &= 6 \pm \frac{2\xi^2}{35} - \frac{4\xi^4}{2940} \pm \frac{352800\pi^2 - 2244421}{15126300000}\xi^6 \\
&\quad + \frac{2972114029 - 2074464000\zeta_3 + 117600\pi^2(1680\log(2) - 1801)}{148237740000000}\xi^8 \\
&\quad + \mathcal{O}(\xi^{10})
\end{aligned}
$$

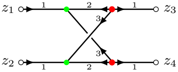
All-loop anomalous dimension in the super brick wall theory

Extract anomalous dimension of $\mathrm{tr}\,[\Phi_1\Phi_1]$ from

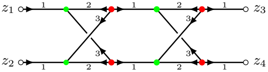
$$\left\langle \mathrm{tr}\,[\Phi_1(z_1)\Phi_1(z_2)]\,\mathrm{tr}\left[\Phi_1^\dagger(z_3)\Phi_1^\dagger(z_4)\right]\right\rangle =$$



$+\,\xi^4$



$+\,\xi^8$



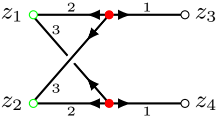
$+\cdots$

$$+\,(z_3 \leftrightarrow z_4)$$

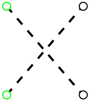
$$\sim \frac{(1+\mathbb{P})}{1-\xi^4\mathbb{H}}$$

with graph-building operators

$\mathbb{H} =$



, $\mathbb{P} =$



,

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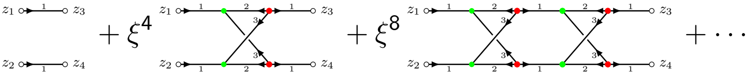


$$+ (z_3 \leftrightarrow z_4)$$

$$\sim \frac{(1+\mathbb{P})}{1-\xi^4\mathbb{H}} = \sum\!\!\!\!\!\!\!\int_{\Delta,S} |\bar{\Omega}_{\Delta,S,2}\rangle\langle\Omega_{\Delta,S,2}|\frac{1}{1-\xi^4 E_0^{\mathrm{SBW}}(\Delta,S)}$$

with a complete set of eigenfunctions $\mathbb{1} = \sum\!\!\!\!\!\!\!\int_{\Delta,S} |\bar{\Omega}_{\Delta,S,R}\rangle\langle\Omega_{\Delta,S,R}|$

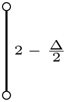
and eigenvalue $\langle\Omega_{\Delta,S,2}|\mathbb{H} = \langle\Omega_{\Delta,S,2}|E_0^{\mathrm{SBW}}(\Delta,S)$. The spinless pole condition is

$$1 = \xi^4 E_0^{\mathrm{SBW}}(\Delta,0)^2$$

All-loop anomalous dimension in the super brick wall theory

Again, I conjecture the limit of the $R = 2$ eigenfunction

$$\Omega_{\Delta,0,2} \overset{\substack{x_0 \to \infty \\ \theta_0, \bar{\theta}_0 = 0}}{\sim} \Psi_{2-\frac{\Delta}{2}} :=$$



All-loop anomalous dimension in the super brick wall theory

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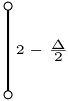
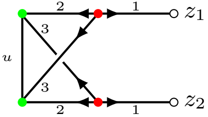
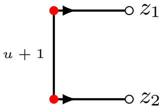


Calculate the eigenvalue of $\mathbb{H}$ from

$$\langle \Psi_u | \mathbb{H} =$$



$$\sim$$



$$\sim \mathrm{E}_0^{\mathrm{SBW}}(u) \langle \Psi_u |$$

All-loop anomalous dimension in the super brick wall theory

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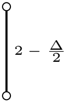

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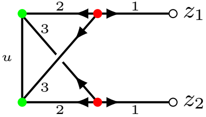
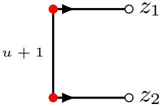


$$\sim$$



$$\sim \mathrm{E}_0^{\mathrm{SBW}}(u) \langle \Psi_u |$$

One finds the **eigenvalue** in agreement with [Kazakov,Olivucci,Preti'19]

$$\mathrm{E}_0^{\mathrm{SBW}}(u) = -\,8\,u \cdot r(2-u, u+1, 1)\, r(2-u, u, 1) \cdot \left[\psi^{(1)}\left(\tfrac{u-1}{2}\right) - \psi^{(1)}\left(\tfrac{1-u}{2}\right) + \psi^{(1)}\left(\tfrac{2-u}{2}\right) - \psi^{(1)}\left(\tfrac{u}{2}\right)\right]$$

Perturbative relaxation of the double-scaling limit

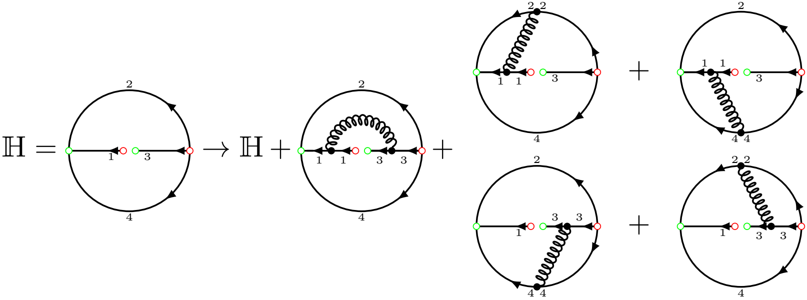
Recall β-deformed ABJM
$S_{\mathrm{ABJM},\beta} = -\mathrm{i}\,\frac{\mathrm{N}}{\lambda^2} \cdot S_{\mathrm{CS}} + \mathrm{N} \cdot S_{\mathrm{mat}} + \mathrm{N}\lambda^2 \cdot S_{\mathrm{pot}}$, replace $q = 2\xi/\lambda^2$ and expand up to λ^2

$$
\begin{aligned}
S_{\mathrm{ABJM},\beta} =& S^{\mathrm{SFN}} + \frac{\mathrm{iN}}{2} \int \mathrm{d}^7 z \ \mathrm{tr}\left[\mathcal{V}\bar{D}D\mathcal{V} + \hat{\mathcal{V}}\bar{D}D\hat{\mathcal{V}}\right] \\
& + \lambda \mathrm{N} \int \mathrm{d}^7 z \ \mathrm{tr}\Big[\Phi_1\Phi_1^\dagger\mathcal{V} + \Phi_3\Phi_3^\dagger\mathcal{V} - \Phi_1^\dagger\Phi_1\hat{\mathcal{V}} - \Phi_3^\dagger\Phi_3\hat{\mathcal{V}} \\
& \qquad - \Phi_2^\dagger\Phi_2\mathcal{V} - \Phi_4^\dagger\Phi_4\mathcal{V} + \Phi_2\Phi_2^\dagger\hat{\mathcal{V}} + \Phi_4\Phi_4^\dagger\hat{\mathcal{V}} \\
& \qquad + \frac{\mathrm{i}}{3}\mathcal{V}\bar{D}(\mathcal{V}D\mathcal{V}) - \frac{\mathrm{i}}{6}\mathcal{V}\bar{D}D\mathcal{V}^2\Big] \\
& + \mathcal{O}\left(\lambda^2\Phi_i^\dagger\mathcal{V}\Phi_i\hat{\mathcal{V}}\right) + \mathcal{O}\left(\lambda^2\Phi_i^\dagger\mathcal{V}^2\Phi_i\right) + \mathcal{O}\left(\lambda^2\mathcal{V}^3\right) + \mathcal{O}\left(\lambda^3\right)
\end{aligned}
$$

Hermitian conjugates of quartic scalar interaction fortunately appear only at $\mathcal{O}(\lambda^4)$

Perturbative relaxation of the double-scaling limit

Consider again the zero-magnon correlator, and its graph-builder

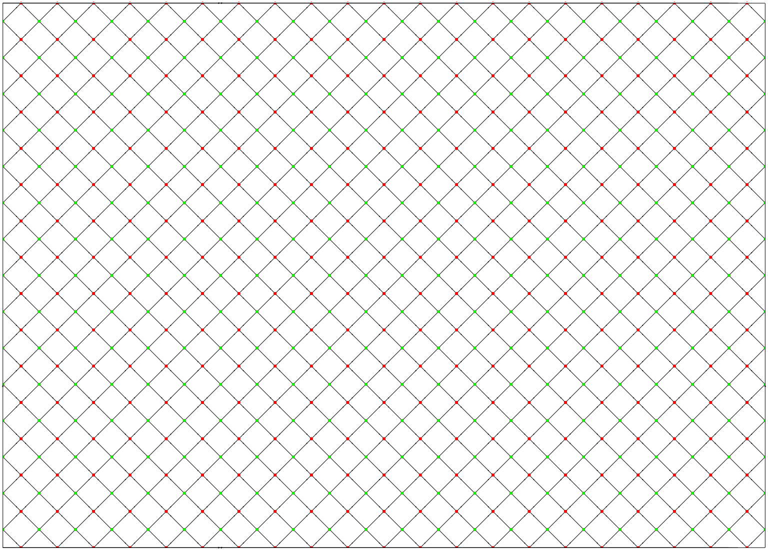


Diagonalization? Chain rules and STR for gauge superpropagator? Choice of eigenfunctions?

Outlook

- Uncovering the complete integrable structure of the super fishnet model (super brick wall) would require the construction of non-compact superconformal $osp(2|4)$ ($s\ell(4|1)$) R-matrices. Are the chain relations and star-3$\times$triangle relation sufficient for this purpose?
- How to uplift further exact results for non-dynamical fishnet models to super brick wall and super fishnet models?
- Can the perturbative relaxation of the double-scaling limit help us to understand how the lattice-like structures evolve into ABJM (or $\mathcal{N} = 4$ SYM) Feynman supergraphs?
- We showed that supergraphs can prevent the 'dynamical melting' of fishnets. Could similar ideas possibly work for general dynamical fishnets, or even for $\mathcal{N} = 4$ SYM? E.g. Harmonic superspace

Thanks for your attention!



Superspace formulation of double-scaled β-deformation of $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ $\mathrm{SU(N)}$ SYM in $\mathcal{N} = 1$ superspace formulation
[Penati,Santambrogio'01]

$$S = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \sum_{i=1}^{3} \mathrm{tr}\left[\mathrm{e}^{-gV}\Phi_i^\dagger \mathrm{e}^{gV}\Phi_i\right] + \frac{1}{2g^2}\int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{tr}\left[W^\alpha W_\alpha\right]$$
$$+ \mathrm{i}g \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{tr}\left[\Phi_1[\Phi_2, \Phi_3]\right] + \mathrm{i}g \int \mathrm{d}^4x \, \mathrm{d}^2\bar{\theta} \, \mathrm{tr}\left[\Phi_1^\dagger[\Phi_2^\dagger, \Phi_3^\dagger]\right]$$

β-deformation:
$\Phi_i \cdot \Phi_j \to \Phi_i \star \Phi_j := \mathrm{e}^{\mathrm{i}\,\det\left(\boldsymbol{\gamma}|\mathbf{q}_i|\mathbf{q}_j\right)}\Phi_i \cdot \Phi_j$
with $\boldsymbol{\gamma} = (\beta, \beta, \beta)$,$\mathbf{q}_1 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$, $\mathbf{q}_2 = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ and $\mathbf{q}_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$ [Lunin,Maldacena'05]

Superspace formulation of double-scaled β-deformation of $\mathcal{N} = 4$ SYM

β-deformed $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ superspace formulation [Jin,Roiban'12]

$$S = \int \mathrm{d}^4 x \, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \sum_{i=1}^{3} \mathrm{tr} \left[\mathrm{e}^{-gV} \Phi_i^\dagger \mathrm{e}^{gV} \Phi_i \right] + \frac{1}{2g^2} \int \mathrm{d}^4 x \, \mathrm{d}^2\theta \, \mathrm{tr} \left[W^\alpha W_\alpha \right]$$
$$+ \mathrm{i}g \int \mathrm{d}^4 x \, \mathrm{d}^2\theta \, \mathrm{tr} \left[q \, \Phi_1 \Phi_2 \Phi_3 - q^{-1} \Phi_1 \Phi_3 \Phi_2 \right] + \mathrm{h.c.}$$

with $q = \mathrm{e}^{\mathrm{i}\beta}$

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β-deformed $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ superspace formulation
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$$+ \mathrm{i}g \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{tr}\left[q\, \Phi_1\Phi_2\Phi_3 - q^{-1}\Phi_1\Phi_3\Phi_2\right] + \mathrm{h.c.}$$

with $q = \mathrm{e}^{\mathrm{i}\beta}$
Then

- 't Hooft limit: Rescale the fields for genus expansion and $g \to 0$ and $\mathrm{N} \to \infty$, while $\lambda = g^2\mathrm{N}$ fixed
- Double-scaling limit: $\lambda \to 0$ and $\beta \to -\mathrm{i}\infty \Rightarrow q \to \infty$, while $\xi := \lambda \cdot q$ fixed [Gürdoğan,Kazakov'15]

Superspace formulation of double-scaled β-deformation of $\mathcal{N} = 4$ SYM

Double-scaled β-deformed $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ superspace formulation [MK,Staudacher'24]

$$
\begin{aligned}
S \ &= \mathrm{N} \int \mathrm{d}^4 x \, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \left\{ \sum_{i=1}^{3} \mathrm{tr} \left[\Phi_i^\dagger \Phi_i \right] \right. \\
&\qquad \left. + \mathrm{i}\xi \cdot \bar{\theta}^2 \, \mathrm{tr} \left[\Phi_1 \Phi_2 \Phi_3 \right] + \mathrm{i}\xi \cdot \theta^2 \, \mathrm{tr} \left[\Phi_1^\dagger \Phi_2^\dagger \Phi_3^\dagger \right] \right\}
\end{aligned}
$$

Superspace formulation of double-scaled β-deformation of $\mathcal{N}=4$ SYM

Double-scaled β-deformed $\mathcal{N}=4$ SYM in $\mathcal{N}=1$ superspace formulation [MK,Staudacher'24]

$$S = \mathrm{N}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\left\{\sum_{i=1}^{3}\mathrm{tr}\left[\Phi_i^\dagger\Phi_i\right] + \mathrm{i}\xi\cdot\bar{\theta}^2\,\mathrm{tr}\left[\Phi_1\Phi_2\Phi_3\right] + \mathrm{i}\xi\cdot\theta^2\,\mathrm{tr}\left[\Phi_1^\dagger\Phi_2^\dagger\Phi_3^\dagger\right]\right\}$$

In components: χ-CFT dynamical fishnet [Gürdoğan,Kazakov'15]

$$\begin{aligned} S = \mathrm{N}\int \mathrm{d}^4x\,\mathrm{tr}\Bigg\{&\sum_{i=1}^{3}\left[\phi_i^\dagger\Box\phi_i - \mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i\right] \\ &+\xi^2\left[\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger + \phi_3\phi_1\phi_3^\dagger\phi_1^\dagger + \phi_2\phi_3\phi_2^\dagger\phi_3^\dagger\right] \\ &-\mathrm{i}\xi\left[\phi_1\psi_2\psi_3 + \phi_2\psi_3\psi_1 + \phi_3\psi_1\psi_2\right] \\ &-\mathrm{i}\xi\left[\phi_1^\dagger\bar{\psi}_2\bar{\psi}_3 + \phi_2^\dagger\bar{\psi}_3\bar{\psi}_1 + \phi_3^\dagger\bar{\psi}_1\bar{\psi}_2\right] + \mathcal{L}_{\mathrm{dt}}\Bigg\}\end{aligned}$$

Useful superintegral relations

Osborn's formula [Osborn'98][Dolan,Osborn'00]

$$\mathrm{i}\int \mathrm{d}^4 x_0\ \mathrm{d}^2\theta_0\,\mathrm{d}^2\bar{\theta}_0\ \delta^{(2)}(\theta_0)\ \frac{1}{\left[x^2_{1\bar{0}}\right]^{u_1}}\frac{1}{\left[x^2_{2\bar{0}}\right]^{u_2}}\frac{1}{\left[x^2_{3\bar{0}}\right]^{u_3}}$$

$$\overset{u_1+u_2+u_3=3}{=} -4\,r(u_1,u_2,u_3)\ \frac{(\theta_{12}\theta_{13})\,x^2_{23,+}+(\theta_{23}\theta_{21})\,x^2_{31,+}+(\theta_{31}\theta_{32})\,x^2_{12,+}}{[x^2_{12,+}]^{2-u_3}[x^2_{23,+}]^{2-u_1}[x^2_{31,+}]^{2-u_2}}$$

with $x^\mu_{ij,+} := x^\mu_{i,+} - x^\mu_{j,+}$, $x^\mu_\pm = x^\mu \pm \mathrm{i}\theta\sigma^\mu\bar{\theta}$

and $r(u_1,u_2,u_3) := \pi^2 a(u_1)a(u_2)a(u_3)$, $a(u) := \frac{\Gamma(2-u)}{\Gamma(u)}$