Workshop: “Fishnets-2025”

University of Southampton, July 14, 2025

I. Fishnet CFT: review and problems

II. Complex matrix models vs LLM gravity for protected correlators of N=4 SYM

Vladimir Kazakov

LPENS
LABORATOIRE DE PHYSIQUE DE L’ÉCOLE NORMALE SUPÉRIEURE

ENS | PSL★ cnrs SORBONNE UNIVERSITÉ Université Paris Cité

Fishnets

Outline

- Fishnet CFT is 10 years old. Discovered as special limit of gamma-deformed N=4 SYM. Unique case of integrable (non-unitary) CFTs in any dimensions.

  Many generalizations ("Loom" fishnet CFT's, Checkerboard CFT). Fishnet-type solvable matrix models

- Many applications:

  Feynman graphs, Yangian symmetry, 2d Calabi-Yau manifolds, non-equilibrium and non-unitary processes in condensed matter (?), going beyond fishnet limit in $\mathcal{N}$=4 SYM, new CFT/string correspondence ("fish-chain" construction).

- Many problems left behind: universal approach to computation of dimensions of various operators in general fishnet CFT's. QSC for fishnets, quantization conditions etc.

- I review only a few of these points…

Dynamical "Fishnet" from N=4 SYM

Gurdogan, V.K. '15

- Double scaling "fishnet" limit in γ-twisted N=4 SYM : Strong imaginary γ-twist, weak coupling:

$$g \to 0, \qquad \gamma \to i\infty, \qquad \xi_j = g\, e^{-i\gamma_j/2} - \text{fixed}, \qquad (j = 1,2,3.)$$

- Chiral CFT from double-scaled N=4 SYM:

$$\mathcal{L} = N_c \mathrm{tr}[-\frac{1}{2}\partial^\mu \bar{\phi}_i \partial_\mu \phi^i + i\bar{\psi}^{\dot{\alpha}}_A \partial^\alpha_{\dot{\alpha}} \psi^A_\alpha] + \mathcal{L}_{\text{int}}$$

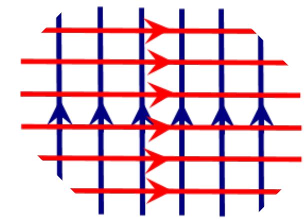
$$\mathcal{L}_{\text{int}} = N_c \,\mathrm{tr}[\xi_1^2\, \bar\phi_2\bar\phi_3\phi_2\phi_3 + \xi_2^2\, \bar\phi_3\bar\phi_1\phi_3\phi_1 + \xi_3^2\, \bar\phi_1\bar\phi_2\phi_1\phi_2 +$$

3 flavors of bosons and fermions

$$+ i\sqrt{\xi_2\xi_3}(\psi^3\phi^1\psi^2 + \bar\psi_3\bar\phi_1\bar\psi_2) + i\sqrt{\xi_1\xi_3}(\psi^1\phi^2\psi^3 + \bar\psi_1\bar\phi_2\bar\psi_3) + i\sqrt{\xi_1\xi_2}(\psi^2\phi^3\psi^1 + \bar\psi_2\bar\phi_3\bar\psi_1)\,].$$

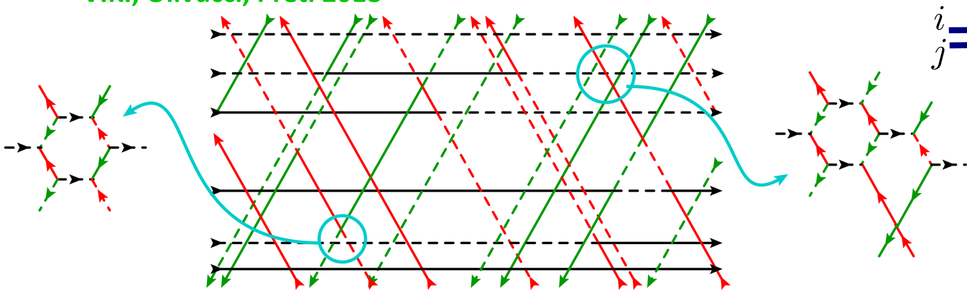
- Planar Feynman graphs form a dynamical fishnet:

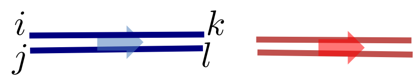
  3 systems of parallel lines, quartic vertices; solid lines – bosons, dashed lines - fermions



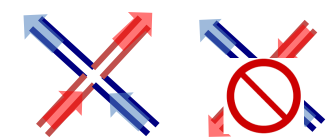
$$\mathcal{L}[\phi_1,\phi_2] = \frac{N_c}{2}\mathrm{tr}\,\left(\partial^\mu\bar\phi_1\partial_\mu\phi_1 + \partial^\mu\bar\phi_2\partial_\mu\phi_2 + 2\xi^2\,\bar\phi_1\bar\phi_2\phi_1\phi_2\right)$$

V.K., Olivucci, Preti 2018





$$\text{propagators} = \frac{\delta_{ik}\,\delta_{jl}}{N_c\,(x-y)^2}$$



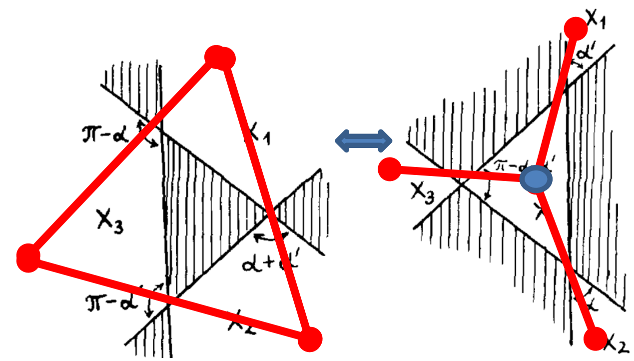
- Inherits N=4 SYM integrability. But what is this integrability?
- Bi-scalar model: explicit integrability!

Gurdogan, V.K. '15
Caetano, Gurdogan, V.K. '16

$$\xi_1 = \xi_2 = \xi_3$$

Kade, Staudacher 2024

- A progress: at susy β-deformation point: "supergraph fishnet"
  Still no Yang-Baxter construction. What about generic couplings?



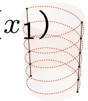
- What are protected operators and correlators in Fishnet CFT at this susy point ?

A. Zamolodchikov 1980

Operators, correlators, graphs…

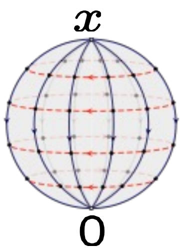
- Explicit computations of correlators

$\mathrm{tr}[\phi_1\phi_2](x_1)$

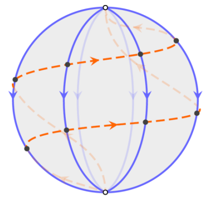


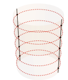
$\mathrm{tr}[\phi_1\phi_2^\dagger](x_2)$

“vacuum” operator $\mathrm{tr}[\phi_1(x)]^L$



Multi-magnon spiral graphs





$\mathrm{tr}[\phi_1^\dagger\phi_2](x_3)$

$\mathrm{tr}[\phi_1^\dagger\phi_2^\dagger](x_4)$

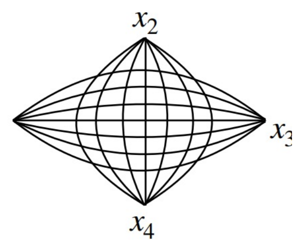
Gurdogan, V.K ‘15
Caetano, Gurdogan, V.K ’16
Gromov, V.K., Korchemsky, Negro, Sizov ‘16
Gromov, Sever ‘20

OPE, 4-point functions, stampedes…

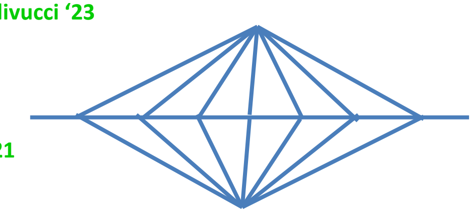
Grabner, Gromov, V.K , Korchemsky ‘17
V.K., Olivucci 2018
Gromov, V.K , Korchemsky ’18
V.K., Olivucci, Preti, ’19
Gromov, Sever ’20
Olivucci, Vieira ’22
Olivucci ‘23

- Basso-Dixon 4-point functions through determinant of “ladder” graphs, Sklyanin SoV

$$G_{m,n}(x_1,x_2,x_3,x_4) \; = \;$$

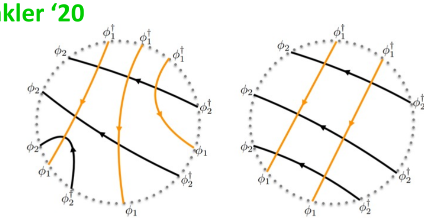


Davidichev, Ushuikina
Basso, Dixon
Derkachev, V.K., Olivucci
Derkachev, Ferrando, Olivucci ‘21
Dercachov, Olivucci ’21
Basso, Dixon, Kosover, Krajenbrink, Zhong ‘21
Kostov ‘23
…Staudacher, Kade ‘23



- Amplitudes, Yangian symmetry, Calabi-Yau periods…

Chicherin, V.K., Mueller, Loebbert, Zheng ’17
Corcoran, Loebbert, Miczajka, Muller, Munkler ‘20
Duhr, Klemm Loebbert, Nega, Porkert’22
V.K., Levkovich-Maslyuk, Mishnyakov ’23
Levkovich-Maslyuk, Mishnyakov ‘25



- Thermodynamical Bethe Ansatz for Fishnet

Basso, Zhong ‘19
Basso, Ferrando, V.K., Zhong ’19

- “Fishchain”: AdS dual for Fishnet

Gromov, Sever ‘19

- Non-trivial flat vacua and spontaneous symmetry breaking in Fishnet CFT

V.K., Karananas, Shaposhnikov ‘19

- Eclectic spin chain from Fishnet

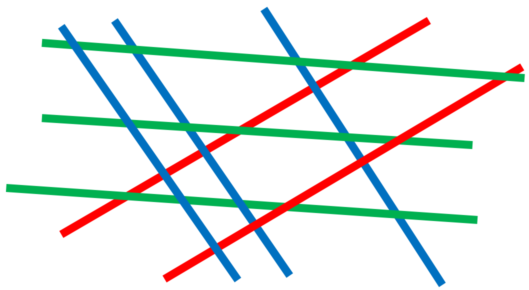
Ahn, Staudacher ‘21, ‘22

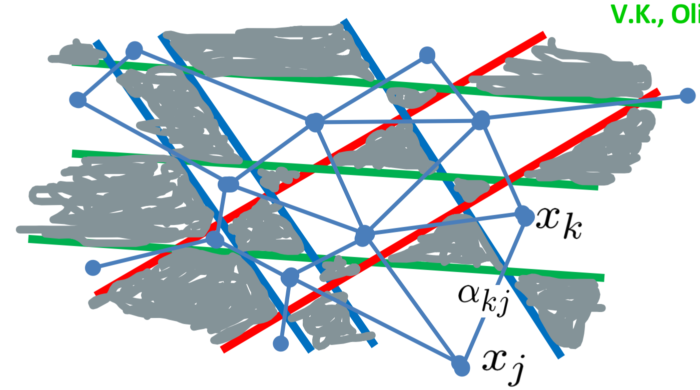
Loom for fishnet CFTs from Baxter lattices

- Baxter lattice for general Fishnet CFT: M intersecting lines with M slopes

  e.g. M=3

V.K., Olivucci 18' '22





$$\mathcal{G}_B = \int \prod_{m\in\mathcal{L}_I} d^D x_m \prod_{<j,k>\in\mathcal{L}_I} G_D(x_j, x_k, \alpha_{jk})$$

$$\boxed{G_D(x_j, x_k, \alpha_{jk}) = |x_j - x_k|^{\frac{D}{\pi}(\alpha_{jk}-\pi)}}$$

Construct Fishnet CFT$^{(M)}$
with all such Feynman graphs
(related by star-triangle)

$$\mathcal{L}_{\mathrm{LOOM}} = \frac{N_c}{2}\mathrm{tr}\left(-\sum_{j=1}^{M(M-1)} \bar{\phi}_j \left(\Box^{D/2-\Delta_{\phi_j}}\right)\phi_j\right) + \mathcal{L}_{int}$$

Checkerboard CFT : Loom with M=4 slopes and only two interaction terms

Alfimov, V.K., Ferrando, Olivucci, '23

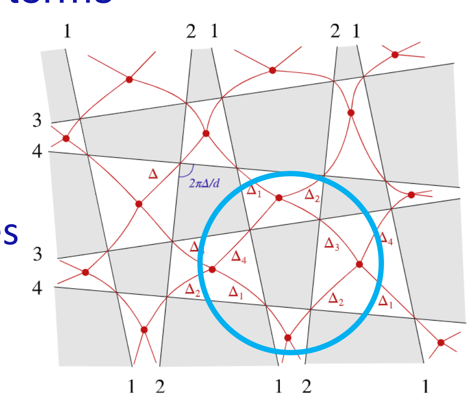
$$\mathcal{L}_D = N_c\mathrm{tr}\left[\underbrace{-\sum_{j=1}^{4}\bar{X}_j\Box^{w_j}X_j}_{\nabla\bar{X}_{\mu,j}\nabla X_j^{\mu}} + \xi_1^2\,\bar{X}_1\bar{X}_2X_3X_4 + \xi_2^2\,X_1X_2\bar{X}_3\bar{X}_4\right],$$

$$\sum_{j=1}^{4} w_j = D$$

If $D=4$
all $w_j = 1$

R-matrix in principals series irrep of conformal group

$$\overset{a}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} = |x_1 - x_2|^{-2a}$$



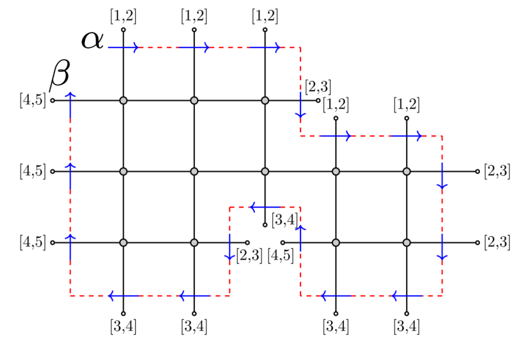
Yangian symmetry for planar correlators

Chicherin, V.K., Loebbert, Muller, Zhong '17 '17

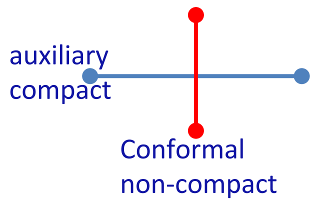
- Single-trace correlator in bi-scalar Fishnet CFT $\rightarrow$ single planar graph

$$\langle \mathrm{Tr}\,[\Phi_1(x_1)\Phi_2(x_2)\ldots\Phi_n(x_n)]\rangle$$

$$\Phi_j \in \{\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2, \}$$



- "Lasso" operator: product of Lax matrices



$$L(u_+, u_-) = \begin{pmatrix} u_+ - \mathbf{px} & \mathbf{p} \\ \mathbf{x}(u_+ - u_-) - \mathbf{xpx} & \mathbf{xp} + u_- \end{pmatrix}$$

$$\mathbf{x} = -i\bar{\sigma}^\mu x_\mu \;, \quad \mathbf{p} = -\frac{i}{2}\sigma^\mu \partial_{x_\mu}$$

$$u_+ = u + \frac{\Delta - D}{2} \;, \quad u_- = u - \frac{\Delta}{2}$$

- Graph is an eigenfunction of Lasso. 1/u- expansion of lasso: Yangian diff.eq.

$$(L_1 L_2 \ldots L_n)_{\alpha\beta}\,|\mathrm{graph}\rangle = \lambda(u)\delta_{\alpha\beta}|\mathrm{graph}\rangle \Longrightarrow \text{PDE}$$

Derkachev, V.K., Ferranc
[Duhr, Klemm Loebbert
Nega, Porkert 22]

- 2d graphs linked with Calabi-Yau geometry

[Corcoran, Loebbert, Miczajka,
Muller, Munkler, Staudacher ,... 18-22

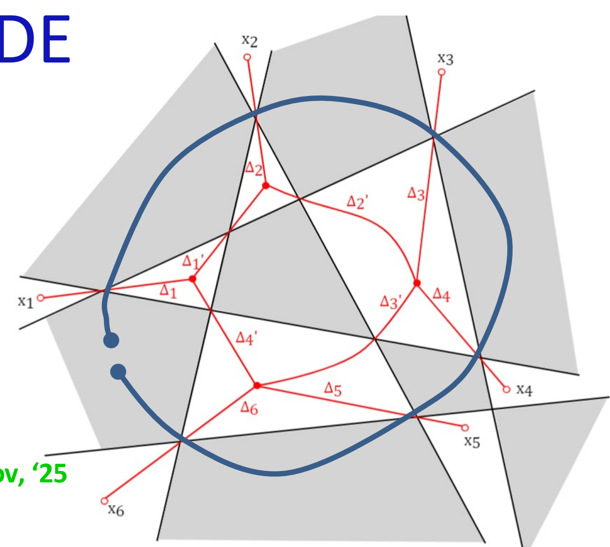
- Generalization this construction from reg. square lattice to arbitrary loom

V.K., Levkovich-Maslyuk, Mishnyakov, '23

- Formulation of Yangian PDE in terms of cross-ratios

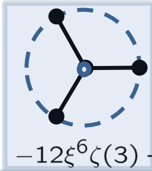
Loebbert, ... 22'...?
Levkovich-Maslyuk, Mishnyakov, '25

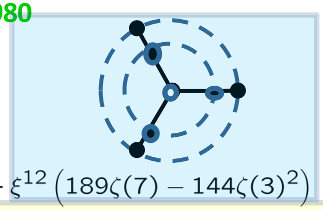
- Lasso construction for cylindric graphs?



Dimension of $\mathrm{tr}(\phi_1)^3$ and periods of wheel graphs from QSC

Broadherst 1980



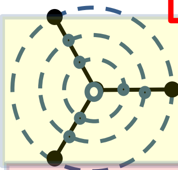


Ahn, Bajnok, Bombardelli, Nepomechie 2013

E.Panzer, 2015

Gurdogan, V.K. '15 (any number of spokes)

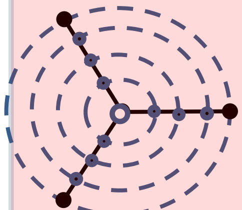
In terms of Riemann (multi)-zeta numbers



$$\boxed{\Delta - 3 =} -12\xi^6\zeta(3) + \xi^{12}\left(189\zeta(7) - 144\zeta(3)^2\right)$$

$$+\xi^{18}\left(-1944\zeta(8,2,1) - 3024\zeta(3)^3 - 3024\zeta(5)\zeta(3)^2 + \frac{198\pi^8\zeta(3)}{175} + 6804\zeta(7)\zeta(3)\right.$$
$$\left.+\frac{612\pi^6\zeta(5)}{35} + 270\pi^4\zeta(7) + 5994\pi^2\zeta(9) - \frac{925911\zeta(11)}{8}\right) +$$

Gromov, V.K , Korchemsky, Negro, Sizov '17



$$\xi^{24}\left(\frac{10368}{5}\pi^4\zeta(8,2,1) + 5184\pi^2\zeta(9,3,1) + 51840\pi^2\zeta(10,2,1) - 148716\zeta(11,3,1)\right.$$
$$-1061910\zeta(12,2,1) + 62208\zeta(10,2,1,1,1) - 93312\zeta(3)\zeta(8,2,1) - 288\zeta(3)^5$$
$$+72\gamma\pi^2\zeta(3)^4 - 77760\zeta(3)^4 - \frac{80756\pi^6\zeta(3)^3}{945} - 145152\zeta(5)\zeta(3)^3 - \frac{29}{270}\gamma\pi^8\zeta(3)^2$$
$$+\frac{9504\pi^8\zeta(3)^2}{175} - 879\pi^4\zeta(5)\zeta(3)^2 - 2025\pi^2\zeta(7)\zeta(3)^2 + 244944\zeta(7)\zeta(3)^2$$
$$+186588\zeta(9)\zeta(3)^2 + \frac{2910394\pi^{12}\zeta(3)}{2627625} - 2592\pi^2\zeta(5)^2\zeta(3) + \frac{29376}{35}\pi^6\zeta(5)\zeta(3)$$
$$+12960\pi^4\zeta(7)\zeta(3) + 298404\zeta(5)\zeta(7)\zeta(3) + 287712\pi^2\zeta(9)\zeta(3)$$
$$-5555466\zeta(11)\zeta(3) + 57672\zeta(5)^3 - 71442\zeta(7)^2 + \frac{13953\pi^{10}\zeta(5)}{1925} + \frac{7293\pi^8\zeta(7)}{175} - \frac{19959\pi^6\zeta(9)}{5}$$
$$\left.+\frac{119979\pi^4\zeta(11)}{2} + \frac{10738413\pi^2\zeta(13)}{2} - \frac{4607294013\zeta(15)}{80}\right) + O\left(\xi^{25}\right)$$

- Based on Quantum Spectral curve of AdS5/CFT4

Gromov, V.K , Leurent, Voilin '13, '14
V.K , Leurent, Voilin '15

Baxter eq.:
$$\left(\frac{(\Delta-1)(\Delta-3)}{4u^2} - \frac{i\,\xi^3}{u^3} - 2\right) q(u) + q(u+i) + q(u-i) = 0$$

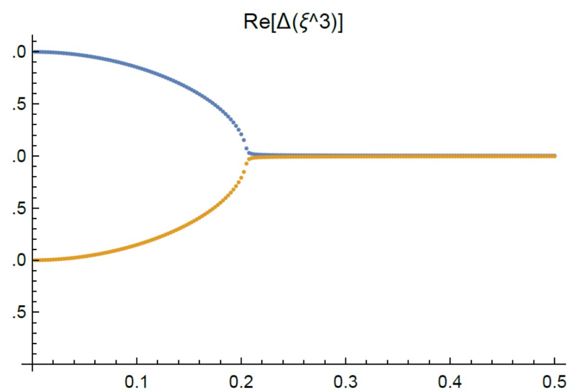
Asymptotics:
$$q_1(u,\xi) \sim u^{\Delta/2-1/2}\left(1 + \frac{\alpha_1}{u} + \frac{\alpha_2}{u^2} + \cdots\right)$$
$$q_2(u,\xi) \sim u^{-\Delta/2+3/2}\left(1 + \frac{\beta_1}{u} + \frac{\beta_2}{u^2} + \cdots\right)$$

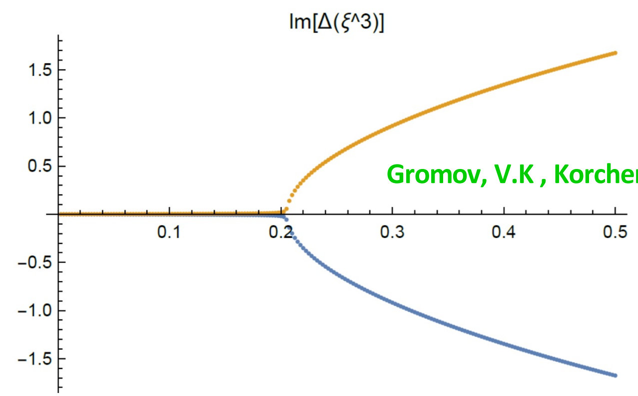
Quantization condition
$$q_1(0,\xi)\, q_2(0,-\xi) + q_2(0,-\xi)\, q_1(0,\xi) = 0$$

- Interesting relation to Galois coaction on Feynman periods Gurdogan'21
- General approach to computation of anomalous dimensions for Fishnet CFT still missing! Gromov, Sever '20, '21

High precision numericsfor spectrum and "PT" symmetry

$\mathrm{tr}(\phi_1)^3$





Gromov, V.K , Korchemsky, Negro, Sizov (2017)

- The two dimensions are real for $\xi < \xi_c$, but they turn to complex conjugates for $\xi > \xi_c$
- The reason for this *reality of spectrum*: "PT" symmetry of Fishnet CFT V.K., Olivucci '22

"PT"-transformation leaves the action invariant (but not operators!):

$$\mathrm{tr}(\phi_1\phi_2\bar{\phi}_1\bar{\phi}_2) \xrightarrow[\text{complex conjugate}]{T} \mathrm{tr}(\phi_2\phi_1\bar{\phi}_2\bar{\phi}_1) \xrightarrow[\text{transpose}]{\text{"}P\text{"}} \mathrm{tr}(\phi_1\phi_2\bar{\phi}_1\bar{\phi}_2)$$

Conformal dimension gets complex conjugate (non-unitary theory!):

$$\left[\langle\bar{\mathcal{O}}(x)\mathcal{O}(0)\rangle\right]^{\mathrm{PT}} = \langle\bar{\mathcal{O}}^{\mathrm{PT}}(x)\mathcal{O}^{\mathrm{PT}}(0)\rangle = |x|^{-2\Delta^*}$$

The spectrum consists of real dimensions or of complex conjugate pairs!

Similar to energy spectrum of non-unitary PT-invariant quantum mechanics

$$\mathcal{H} = \hat{p}^2/2 + x^2\,(ix)^{\epsilon}$$

Bender & Boettcher '98

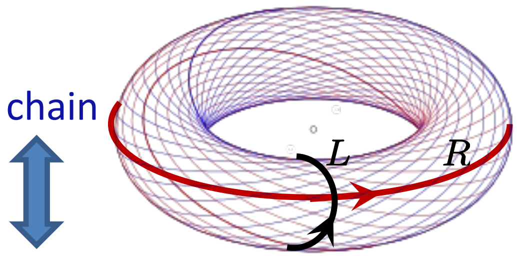
- Applications to non-equilibrium and non-unitary processes in condensed matter

Basso, Zhong '18 (D=4)
Basso, Ferrando, V.K., Zhong (any D)

TBA for Fishnet CFT: knowing S-matrix and dispersion relation we use Yang-Yang eqs. + Al.Zamolodchikov trick for torus partition function

mirror spin chain

double Wick rotation $L \Longleftrightarrow R$



physical spin chain

$$Z = \sum_{E_L} e^{-R\,E_L} = \sum_{E_R} e^{-L\,E_R} \underset{R\to\infty}{\Longleftrightarrow} e^{-R\,E_L^{\text{vac}}}$$

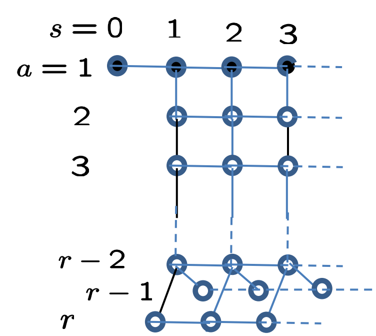
- From TBA to Y-system: anom. dimensions

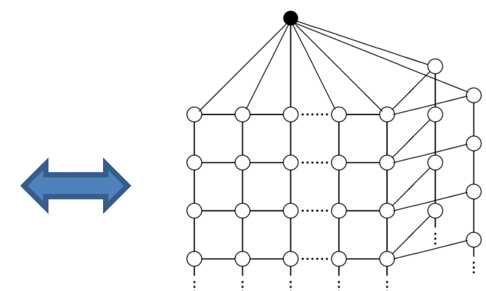
$$\gamma = -2\sum_{s\geq 0}\int \log(1+Y_{1,s})\frac{du}{2\pi}$$

$$Y_{1,0}^{[r-1]}Y_{1,0}^{[1-r]} = \frac{\prod_{k=1}^{r-2}(1+1/Y_{r-k-1,1}^{[k]})(1+1/Y_{r-k-1,1}^{[-k]})}{(1+1/Y_{r-1,1})(1+1/Y_{r,1})}$$

$$\frac{Y_{a,s}^{+}Y_{a,s}^{-}}{Y_{a,s+1}Y_{a,s-1}} = \frac{\prod_{b=1}^{r}(1+Y_{b,s})^{I_{ab}}}{(1+Y_{a,s+1})(1+Y_{a,s-1})}$$

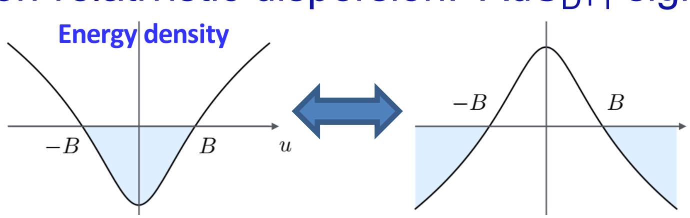
$$1 \leq a \leq r,\ \ s \geq 0$$

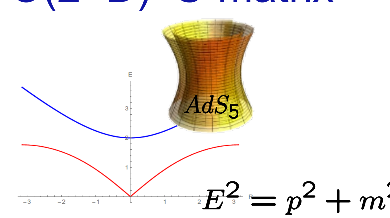




Balog, Hegedus
Frassec, V.K., Ferrando

- Particle-hole duality: we get a sigma-model with Zamolodchikov O(2+D) S-matrix but with with non-relativistic dispersion: AdS_{D+1} sigma model?





$$E^2 = p^2 + m^2$$

- Aymptotics of large fishnet graph $\sim \xi_c^{\text{Area}}$

A. Zamolodchikov 1980

$$\log \xi_c^2 = \int_0^\infty \left[\frac{D}{2}e^{-t} + \frac{e^{-\delta t} - e^{\delta t} + e^{-\tilde{\delta} t} - e^{\tilde{\delta} t}}{(1-e^{-t})(1+e^{\frac{Dt}{2}})}\right]\frac{dt}{t}$$

$$\sinh^2\frac{E}{2} = \tan^2\left(\frac{\pi\delta}{D}\right)\times\sin^2\frac{P}{2}$$

- Quantum spectral curve for D-dimensional fishnet CFT?

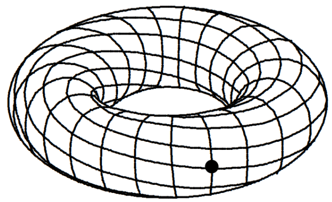
Fishnet matrix models for discrete spins

- Matrix model for "empty" fishnet

Kostov, Staudacher '96

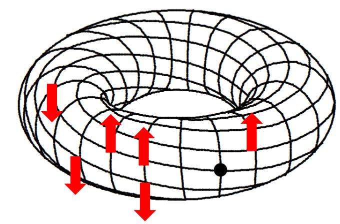
$$Z_{\text{empty fishnet}} = \int d^2X\, d^2Z \;\exp N\mathrm{Tr}\Big[-\bar{X}\,X - \bar{Z}\,Z + \xi^2\, X\,Z\,\bar{X}\,\bar{Z}\Big]$$

$$\log Z_{\text{empty fishnet}} = N^0 \ln\Big[\prod_{n=1}^{\infty}(1-q^{2n})\Big] = \sum$$



$$= q^{-\frac{1}{12}}\,\eta\left(\frac{\log q}{2\pi i}\right) \qquad q \sim \xi$$

- Proposal: to study "fishnet" matrix models generating usual statistical mechanics of discrete spins.
  Example: Ising model on regular graphs (topology of torus):



$$Z_{\text{Ising}} = \int \prod_{c=1,2} d^2X_c\, d^2Z_c \;\exp N\mathrm{Tr}\Big[-\bar{X}_a\Delta^{ab}X_b - \bar{Z}_a\Delta^{ab}Z_b + g\sum_{a=1,2} X_a Z_a \bar{X}_a \bar{Z}_a\Big]$$

Free fermions. Result is known

Kaufman'49

$$\Delta^{a,b} = \begin{pmatrix} 1 & -e^{\beta} \\ -e^{\beta} & 1 \end{pmatrix}$$

- What is the matrix transformation to free fermions? Kramers-Wannier duality?
- Other models? 6-vertex, O(n), BKT, Potts… Can we solve them by matrix methods?
- More complicated boundary conditions and correlators
- New solvable spin models?

(Un)solvable matrix models for N=4 SYM BPS correlators and their gravity duals

- Classifiction of operators by size of dimensions:
  - Small: $\Delta \sim N^0$ (planar integrability)
  - Giant $\Delta \sim N$ (giant gravitons…)
  - Huge $\Delta \sim N^2$ (multi-gravitons, black holes)

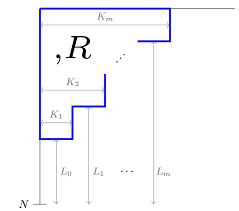
- Protected BPS operators :
  - ½- ¼- and 1/8 BPS (multi-gravitons, entropy ~ N)
  - 1/16-BPS (black holes, entropy ~ N^2)

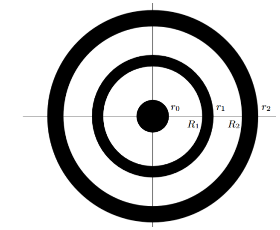
1/2BPS vs LLM metric

Lin, Lunin, Maldacena '04

- 1/2BPS operators: any invariant function of a chiral scalar field $\left(y\cdot\phi\left(x\right)\right)\,,\qquad y\cdot y=0$

- GL(N) Character : $O_{\Delta,R}(x,y)=\chi_R\left(y\cdot\phi\left(x\right)\right)$



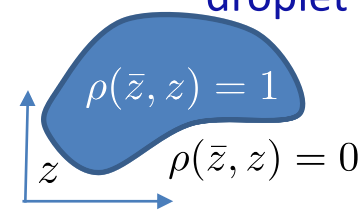


- Coherent state : $O_{CS}(Z,\Lambda)=\int[dU]_{U(N)}e^{\mathrm{Tr}[UZU^\dagger\Lambda]}$

$Z=\phi_5+i\phi_6$

droplet

- Exponential (orthogonal) : $O_{\boldsymbol{t}}(Z)=\exp\left[\sum_{k>0}\frac{t_k}{k}\mathrm{tr}[Z^k]\right]=e^{\mathrm{tr}[V_{\boldsymbol{t}}(Z)]}$

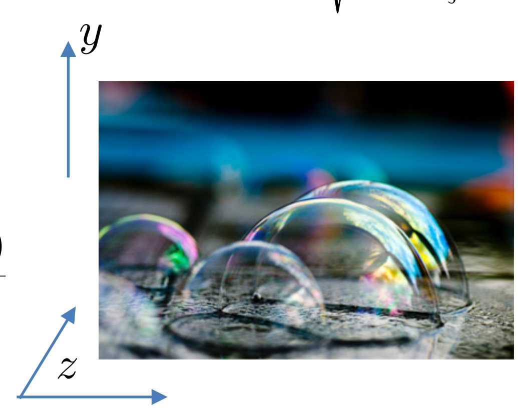


- 1/2BPS states in IIB sugra described by LLM metric: $R\times SO(4)\times SO(4)$

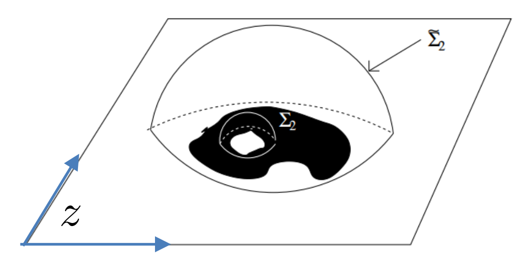
$$ds^2=-\frac{y}{\sqrt{\frac14-\zeta^2}}(dt+\bar V dz+Vd\bar z)^2+\frac{\sqrt{\frac14-\zeta^2}}{y}(dy^2+d\bar z dz)+y\sqrt{\frac{1+2\zeta}{1-2\zeta}}d\Omega_3+y\sqrt{\frac{1-2\zeta}{1+2\zeta}}d\tilde\Omega_3$$

$$\zeta(\bar z,z,y)=\frac{y^2}{\pi}\int_{\mathbb{C}}\frac{\left(\rho(\bar z,z)-\frac12\right)\,d^2z'}{[(\bar z-\bar z')(z-z')+y^2]^2}$$

$$V(\bar z,z,y)=\frac{1}{2\pi}\int_{\mathcal{D}}\frac{(\rho(\bar z,z)-\frac12)\,d\bar z'\,(z-z')}{[(\bar z-\bar z')(z-z')+y^2]^2}$$



bubbling geometry



Anemp odistov, Holguin, V.K., Murali (to appear)

BPS correlators in N=4 SYM

- No YM coupling dependence! Gaussian YM functional integral replaced by gaussian complex matrix measure

$$\langle O_1(x_1)O_2(x_2)\ldots\rangle = \frac{1}{\mathcal{Z}}\int \prod_{I=1}^{6} D\Phi_I \; e^{-N\int \frac{d^4x}{(2\pi)^4}\mathrm{tr}\partial_\mu\Phi^I(x)\partial^\mu\Phi_I(x)}\Big(O_1(x_1)O_2(x_2)\ldots\Big)$$

$$\Longrightarrow \quad \int d^2X\, d^2Y\, d^2Z\; e^{-N\mathrm{tr}\left[X\bar{X}+Y\bar{Y}+Z\bar{Z}\right]}O_1(X,Y,Z)O_2(\bar{X},\bar{Y},\bar{Z})\ldots$$

- 1/2BPS 2-point correlator

$$\langle O\;\bar{O}\rangle = \frac{1}{\mathcal{Z}_\bullet}\int dZ d\bar{Z}\; e^{-N\,\mathrm{tr}[Z\bar{Z}]}\,O(Z)\;\bar{O}(\bar{Z})$$

- 1/2BPS 3-point correlator

$$\langle O\;\bar{O}\;\mathrm{probe}\rangle = \frac{1}{\mathcal{Z}_\bullet}\int dZ d\bar{Z}\; e^{-N\,\mathrm{tr}[Z\bar{Z}]}\,O(Z)\;\bar{O}(\bar{Z})\;F_{\mathrm{probe}}(Z+\bar{Z})$$

$$F_{\mathrm{probe}}(X) = \frac{1}{\mathcal{Z}_M}\int dM\; e^{-\frac{N}{2}\,\mathrm{tr}M^2}\mathrm{probe}\left(X+\frac{iM}{2}\right)$$

Anempodistov, Holguin, V.K., Murali (to appear)

Eigenvalue representation for ½-BPS

- Schur decomposition $Z = U(z+T)U^\dagger$

- 2-point correlator of exponential operators (Ginibre ensemble)

$$\langle O_{V_t}\, O_{\bar V_{\bar t}} \rangle = \int_{\mathbb{C}} \prod_{j=1}^{N} d^2 z\, e^{-|z_j|^2 + V_{\boldsymbol t}(z_j) + \bar V_{\bar{\boldsymbol t}}(\bar z_j)} |\Delta(z)|^2$$

- Saddle point equation (Laplacian growth problem!)

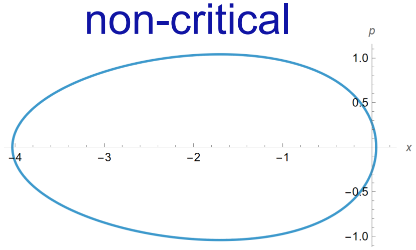
Kostov, Krichever, Mineev, Wigmann, Zabrodin
V.K., Marshakov

$$(\partial_z\, \partial_{\bar z})^{-1} \rho(z,\bar z) = z\bar z - V_{\boldsymbol t}(z) - \bar V_{\boldsymbol t}(z)$$

- Solution – constant density of e.v.'s in a spot defined by potentials: dual to LLM droplet!

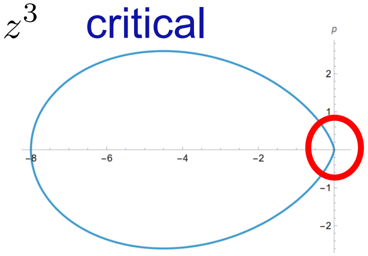
$$\rho(\bar z, z) = \begin{cases} 1 & , z \in \mathcal{D} \\ 0 & , z \notin \mathcal{D} \end{cases}$$

non-critical



$V_{\boldsymbol t}(z) = t_2 z^2 + t_3 z^3$

critical



cusp

pure 2d gravity: $\log\langle O\, O\rangle \sim (t_* - t)^{5/2}$ V.K. '85, David '85

Ising + 2d gravity: $\log\langle O\, O\rangle \sim (t_* - t)^{7/3}$ V.K. '86

- Double scaling limit (Painleve I, KdV, …): 2d QG → 10 SUGRA quantum effects !
- Advantage of complex matrices: 2D e.v. distribution is exactly dual to the LLM droplet !

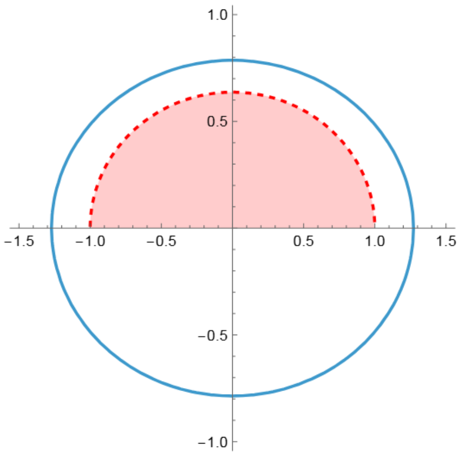
1/2BPS coherent state correlator

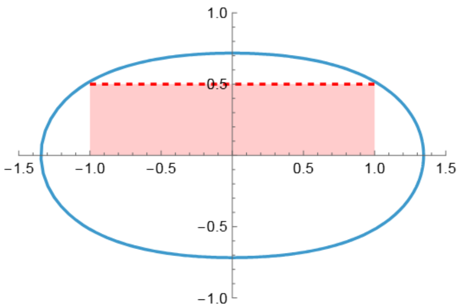
Anempodistov, Holguin, V.K., Murali (to appear)

- 1/2BPS CS correlator reduces to Charish-Chandra-Itsykson-Zuber (CHIZ) integral:

$$\mathcal{Z}_{CS} = \int d^2Z\, dU\, dV\, e^{N\,\mathrm{tr}\left[-Z\bar{Z}+\bar{\Lambda}UZU^\dagger+\Lambda V\bar{Z}V^\dagger\right]} = \int dU e^{N\,\mathrm{tr}U\Lambda U^\dagger\bar{\Lambda}} = \frac{\det_{ij} e^{N\lambda_i\bar{\lambda}_j}}{\Delta(\lambda)\Delta(\bar{\lambda})}$$

- But what is the distribution of e.v.'s ***z_j*** (droplet shape!) ?
  We computed it for semi-circle and uniform distributions





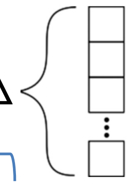
Parametric representation
for the boundary of the spot (droplet)

$$z(\theta) = a + \frac{b-a}{|b-a|^2}\log\left(e^{|b-a|^2} + e^{i\theta}\sqrt{e^{2|b-a|^2} - e^{|b-a|^2}}\right)$$

HH-Giant graviton (HHGg) correlators

Anempodistov, Holguin, V.K., Murali (to appear)

- HHGg correlator looks like an instanton in matrix models:

Δ  $O_{\rm Gg} = \chi_\Delta(Z+\bar Z)$

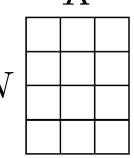
$$C_{R\,R'\,\Delta} = \frac{1}{\sqrt{\mathcal{N}_\Delta}} \overbrace{\frac{(\sqrt{N})^{N-\Delta}}{(N-\Delta)!}\,\frac{1}{\mathcal{Z}_\sigma}\int d\sigma\, e^{-\frac{N}{2}\sigma^2}\,\mathrm{He}_{N-\Delta}(\sqrt{N}\sigma)}^{\text{Gg operator}}$$

$$\times\frac{1}{\sqrt{\mathcal{Z}_R\mathcal{Z}_{R'}}}\underbrace{\int\prod_k |dz_k|^2\, e^{-Nz_k\bar z_k}\,(z_k+\bar z_k+\sigma)\det\left(z_j^{h_l}\right)\det\left(\bar z_n^{h'_m}\right)}_{\text{HH background}}.$$

- Saddle point calculation w.r.t. σ : Gg in general HH background

$$C_{\bar H H,\Delta}\sim\int d\sigma\, e^{-\frac{N}{2}\sigma^2}\,\mathrm{He}_{N-\Delta}(\sqrt{N}\sigma)\,\exp\left[N\int dz d\bar z\,\rho_O(z,\bar z)\log(z+\bar z+\sigma)\right]$$

K

N 

- Explicit result for K × N rectangular YT

$$C_{K\,K\,\Delta} = (-1)^{\Delta/2}\sqrt{\frac{(N-\Delta)!}{N!}}\times\frac{(K+\Delta/2-1)!}{(K-1)!}\,\frac{(N-\Delta/2)!}{(N-\Delta)!(\Delta/2)!}\simeq(-1)^{\frac{\Delta}{2}}\left[\frac{\left(1-\frac{\delta}{2}\right)^{1-\frac{\delta}{2}}\left(\gamma+\frac{\delta}{2}\right)^{\gamma+\frac{\delta}{2}}}{(1-\delta)^{\frac{1-\delta}{2}}\,\gamma^\gamma\left(\frac{\delta}{2}\right)^{\frac{\delta}{2}}}\right]^N$$

$$\gamma = K/N,\qquad \delta=\Delta/N$$

- Challenge: to reproduce it in SUGRA from configuration LLM metric + D-brain

Heavy-Heavy-Heavy (HHH) correlator

V.K., Murali, Vieira '24

- General approach to HHH correlators of character operators through (1+1) fluid dynamics
- Some HHH correlators are computed explicitly: e.g. for 3 rectangular or 3 triangular YT's

Anempodistov, Holguin, V.K., Murali (to appear)

- HHH correlator of 3 exponential operators (related to 2d gravity + 3-state Potts) $O(M) = e^{N\,\mathrm{tr}\,(-t_2 M^2 + t_3 M^3)}$

V.K. '88
V.K., Kostov '91
Daul '94
Eynard, Kristijansen '94

$$\langle\, O\, O\, O\,\rangle = \int \prod_{i=1}^{3} dM_i\; e^{N\,\mathrm{tr}\left[-\sum_i \frac{M_i^2}{2} + \sum_{i<j} M_i M_j\right]}\, O(M_1)\, O(M_2)\, O(M_3)$$

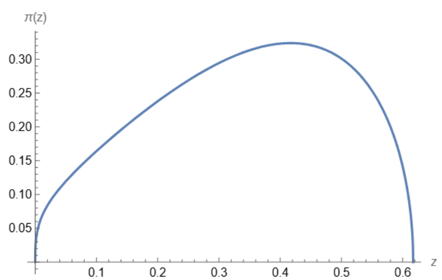
$$= \int dX\, e^{-\frac{N}{2}\mathrm{tr} X^2} \left(\int dM\; e^{N\,\mathrm{tr}\left[X\,M - \frac{4+t_2}{2} M^2 + t_3\, M^3\right]} \right)^3$$

- Exact saddle point solution Huge operators ~ exp N^2

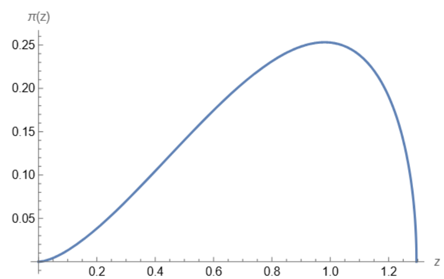
$$\pi(z) = \frac{\sqrt{3}}{2\pi}\left[\left(\frac{B}{z} + \frac{\sqrt{(B-z)(z+B)}}{z}\right)^{-2/3}\left(\frac{z}{\sqrt{3}\alpha}\sqrt{(B-z)(z+B)} + Bb_2 z\right) - \right.$$
$$\left. - \left(\frac{B}{z} - \frac{\sqrt{(B-z)(z+B)}}{z}\right)^{-2/3}\left(-\frac{z}{\sqrt{3}\alpha}\sqrt{(B-z)(z+B)} + Bb_2 z\right)\right]. \quad (1)$$

density

$$z = \sqrt{\alpha x + c}$$



(a) The density $\pi(z)$ on the critical line outside of the critical point. Here $\alpha = 0.1$.



(b) The density $\pi(z)$ at the critical point. Here $\alpha = \alpha_c$.

Figure 13: Comparison of the density of states $\pi(z)$ on the critical line outside and at the critical point.

$\log\langle\, O\, O\, O\,\rangle \sim (g_{cr} - g)^{5/2} + \mathrm{reg.}(g)$ pure 2d gravity

$\log\langle\, O\, O\, O\,\rangle \sim (g_{cr} - g)^{11/5} + \mathrm{reg.}(g)$ 2d gravity+ 3-state Potts, $c = 4/5$

- SUGRA metric for HHH? Generalization of LLM…

SYM/sugra correspondence for HHL correlators

- Skenderis and Taylor computed a few HHL correlators from LLM supergravity, for generic for Little operators (up to charge=4). They matched them successfully to some (extremal) Little operators on SYM side.
- We matched HHL for all such Little operators on SYM side to their sugra results:
- Subtlety: to get correct normalization

- The class of probes that we consider are the $SO(4)_R$ invariant single trace primaries of dimension Δ and charge k, e.g.

$$O_{2,0}(x)=\sqrt{\frac{2}{3}}\mathrm{tr}[Z\bar{Z}-\frac{1}{2}\sum_{i=1}^{4}\phi_i^2]\,,\qquad O_{2,2}(x)=\frac{1}{\sqrt{2}}\mathrm{tr}[Z^2]\,,\qquad O_{3,1}(x)=\frac{1}{\sqrt{2}}\mathrm{tr}[Z(Z\bar{Z}-\sum_{i=1}^{4}\phi_i^2)]$$

Skenderis, Taylor '07

$$\begin{aligned}
\langle O_{\Delta,\Delta}\rangle &= \frac{N^2}{\sqrt{\Delta}\pi^2}(\Delta-2)\sqrt{\Delta-1}\int dzd\bar{z}\rho(z,\bar{z})z^{\Delta},\\
\langle O_{2,0}\rangle &= \frac{N^2}{\pi^2}\frac{\sqrt{2}}{\sqrt{3}}\int dzd\bar{z}\rho(z,\bar{z})\left(z\bar{z}-\frac{1}{2}\right),\\
\langle O_{3,1}\rangle &= \frac{N^2}{\pi^2}\int dzd\bar{z}\rho(z,\bar{z})z^2\bar{z},\\
\langle O_{4,0}\rangle &= \frac{N^2}{\pi^2}\frac{\sqrt{3}}{\sqrt{5}}\int dzd\bar{z}\rho(z,\bar{z})\left(3(z\bar{z})^2-4z\bar{z}+1\right),\\
\langle O_{4,2}\rangle &= \frac{N^2}{\pi^2}\frac{4\sqrt{3}}{\sqrt{10}}\int dzd\bar{z}\rho(z,\bar{z})\,z^2\,(z\bar{z}-1)
\end{aligned}$$

Anempodistov, Holguin, V.K., Murali (to appear)

$$\begin{aligned}
\tau_{\Delta,\Delta} &= \frac{1}{\sqrt{\Delta}}\mathrm{tr}[z^{\Delta}]\to\frac{N}{\sqrt{\Delta}}\int dzd\bar{z}\,\rho(z,\bar{z})\,z^{\Delta},\\
\tau_{2,0} &= \mathrm{N}_{2,0}\mathrm{tr}\left[z\bar{z}-\frac{1}{2}\left(1+\frac{1}{N}\right)\mathbb{I}\right]\to\sqrt{\frac{2}{3}}N\int dzd\bar{z}\rho(z,\bar{z})\left(z\bar{z}-\frac{1}{2}\right),\\
\tau_{3,1} &= \mathrm{N}_{3,1}\mathrm{tr}\left[z^2\bar{z}-\left(1+\frac{1}{N}\right)z\right]\to\frac{N}{\sqrt{2}}\int dzd\bar{z}\rho(z,\bar{z})\left(z^2\bar{z}-z\right),\\
\tau_{4,0} &= \mathrm{N}_{4,0}\mathrm{tr}\left[3(z\bar{z})^2-4\left(1+\frac{2}{3N}\right)z\bar{z}+\left(1+\frac{1}{N}\right)\left(1+\frac{3}{N}\right)\mathbb{I}\right]-\frac{4}{N}(\mathrm{tr}z)(\mathrm{tr}\bar{z}),\\
&\to\frac{N}{2\sqrt{5}}\int dzd\bar{z}\,\rho(z,\bar{z})\left(3(z\bar{z})^2-4z\bar{z}+1\right)\\
\tau_{4,2} &= \mathrm{N}_{4,2}\mathrm{tr}\left[z^3\bar{z}-\left(1+\frac{2}{3N}\right)z^2\right]-\frac{2}{N}(\mathrm{tr}z)^2\to\sqrt{\frac{2}{5}}N\int dzd\bar{z}\,\rho(z,\bar{z})z^2\,(z\bar{z}-1)
\end{aligned}$$

- Perfect match if we fix normalization by comparing the coefficients of chiral primary at Δ=4

$$\langle O_{\Delta,k}\rangle_{\text{gravity}}=\mathcal{N}_k^{\text{sugra}}\times\langle\tau_{\Delta,k}\rangle_{MM}$$
$$\mathcal{N}_k^{\text{sugra}}=\frac{N}{\pi^2}(k-2+\delta_{k,2})\sqrt{k-1}$$

¼- and 1/8-BPS coherent state operators

Anempodistov, Holguin, V.K., Murali (to appear)

- 1/8-BPS coherent state operators are given by unitary integral

$$O_P = \int [dU]_{SU(N)} \, e^{\mathrm{tr}[U^\dagger P^\mu U X_\mu]}, \qquad \mu = 1,2,3$$

Berenstein, Wang '22

- Function of traces of "words" made of complex matrices $\{X_1, X_2, X_3\} = \{X, Y, Z\}$ and of traces of various products of parameters P_μ

- Pair correlator is a (difficult!) unitary integral

$$\langle O_P \bar{O}_{\bar{P}} \rangle = \int [dg] \; e^{\mathrm{tr}[g^\dagger \bar{P}^\mu g P_\mu]}, \qquad g \in SU(N)$$

- Protected 3-point correlator of such ¼-BPS with two exponential ½-BPS operators and is a (difficult!) complex matrix integral in external matrix fields $[P, \bar{P}] = 0$

$$\langle O_V \, O_{\bar{V}} \, O_P \rangle = \int d^{2N^2} Z \, e^{\mathrm{tr}[-\bar{Z}Z + V(Z) + \bar{V}(\bar{Z}) + PZ + \bar{P}\bar{Z}]}$$

- Explicitly calculable case: Quadratic potential (dual to LLM with elliptic droplet) $V(Z) = \nu Z^2$

$$\langle O_V \, O_{\bar{V}} \, O_P \rangle \sim \exp \mathrm{tr} \Big[\frac{-\bar{t}P^2 - t\bar{P}^2 - 2P\bar{P}}{2\,(t\bar{t} - 1)} \Big]$$

¼-BPS CS correlator as Eguchi-Kawai reduction of principal chiral model

Anempodistov, Holguin, V.K., Murali (to appear)

- Two-point correlator of ¼- or 1/8-BPS coherent state operators coincides with partition function of 2D and 3D principal chiral model (PCM) after quenched Eguchi-Kawai (EK) dimensional reduction

$$Z_{\text{PCM}} = \int [\mathcal{D}g]_{SU(N)}\; e^{-\frac{1}{2\lambda}\int d^D x\, \mathrm{tr}[\partial_\mu g \partial^\mu g^\dagger]}$$

$$P^\mu = \mathrm{diag}\{p_1^\mu, \ldots, p_N^\mu\}$$

uniform distribution of quenched momenta in a box $0 < p_j^\mu < \Lambda$

$$\Big\Downarrow\; g(x) \to e^{-iP\cdot x} g e^{iP\cdot x}$$

$$Z_{\text{red}} = \int [dg]\; e^{\frac{V_D}{2\lambda}\mathrm{tr}[P^\mu, g][P_\mu, g^\dagger]} = \int [dg]\;\exp\left(\frac{V_D}{2\lambda}\sum_{k,j=1}^{K}(p^k - p^j)^2 |g_{kj}|^2\right)$$

- Monte-Carlo simulation at N~300 shows that Z_N symmetry is unbroken and the physical asymptotically free behavior is not realized

- Possible fix: twisted Eguchi-Kawai reduction. Uses q-commuting matrices

$$Z = \int [dg]_{SU(N)}\,\exp\left(-\frac{1}{\lambda}\mathrm{tr}[\Gamma_\mu, g][\Gamma_\mu^\dagger, g^\dagger]\right), \qquad [\Gamma_1, \Gamma_2]_q = 0, \qquad q = e^{i\frac{\pi k}{N}}$$

- The following operator in β-deformed SYM is protected at least at one loop

$$O_q = \int [dG]_{SU(N)}\, e^{N\,\mathrm{tr}(G^\dagger P G X + G^\dagger Q G Z)}$$

Dilation operator

$$\mathcal{D}^q_{1-loop}\, O_q = 0$$

- If it is protected in all loops its correlator would be equal to EK reduced twisted PCM !
- PCM is integrable model! Possibility to study ¼-BPS via PCM (and vice versa…)


Thank you