Gauge theories meet enumerative combinatorics

Gregory Korchemsky
IPhT, Saclay

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Fishnet QFTs: Integrability periods and beyond,

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Motivation

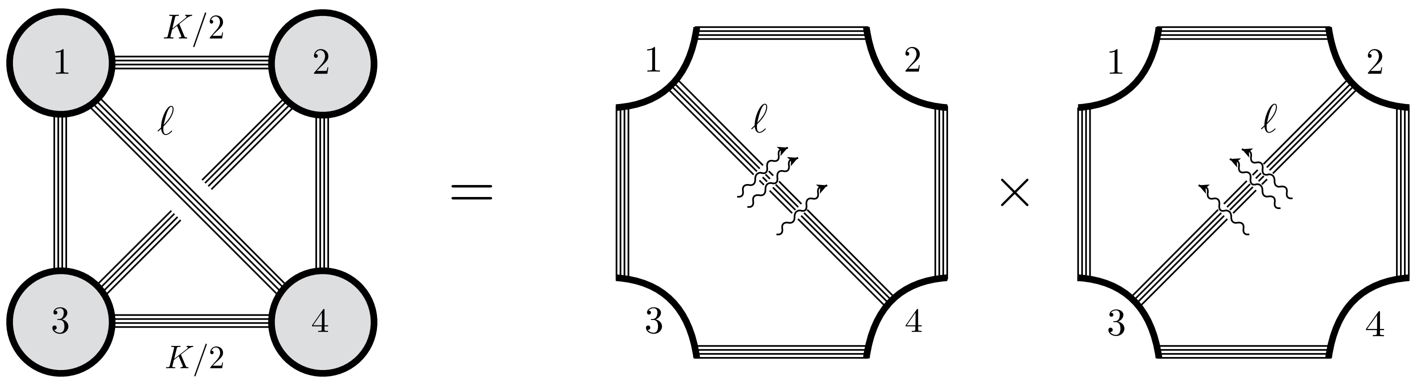
- ✔ Computing observables in four-dimensional supersymmetric theories in the planar limit for finite ’t Hooft coupling remains a challenging task
- ✔ Powerful techniques (localization, integrability) have been developed but a complete solution is still out of reach
- ✔ However, there exists a broad class of observables in four-dimensional $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal Yang–Mills theories for which the problem becomes tractable
- ✔ These observables admit a unified representation as Fredholm determinants of integrable Bessel-type operators

The goal of this talk is to explain connections between four (seemingly unrelated) areas:

- ✗ Solution of planar gauge theories at arbitrary ’t Hooft coupling
- ✗ Fredholm determinants of integrable kernels
- ✗ Iterated Chen integrals and motivic periods
- ✗ Enumerative combinatorics of Dyck paths

Correlation functions in $\mathcal{N} = 4$ SYM

Four-point functions of half-BPS operators $O = \mathrm{tr}(Z^{K/2}X^{K/2}) +$ permutations



Limit of infinitely heavy operators

$$\lim_{K\to\infty} \mathcal{G}_K = \sum_{\ell=\text{bridge length}} [\mathbb{O}_\ell(z,\bar z)]^2$$

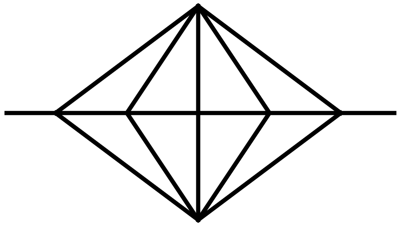
$\mathbb{O}(z,\bar z) =$ 'octagon' is a multilinear combination of ladder integrals [Coronado]

$$\mathbb{O}_{\ell=0}(z,\bar z) = 1 + g^2 f_1 - 2g^4 f_2 + 6g^6 f_3 + g^8(-20 f_4 - \tfrac{1}{2} f_2^2 + f_1 f_3) + \ldots$$

$$= 1 + \sum_{n\geq 1} (g^2)^n \times \sum_{i_1+\cdots+i_m=n} d_{i_1\ldots i_m} f_{i_1}\ldots f_{i_m}$$

From fishnets to Fredholm determinants

Ladder/fishnet integrals

$f_L =$ 

$$= \frac{1}{z-\bar z}\sum_{m=0}^{L}\frac{(-1)^m(2L-m)!}{L!(L-m)!m!}\ln^m(z\bar z)\Big[\underbrace{\mathrm{Li}_{2L-m}(z)}_{\text{polylog}}-\mathrm{Li}_{2L-m}(\bar z)\Big]$$

The octagon admits an *exact* representation as a determinant of a semi-infinite (Bessel) matrix [Kostov,Petkova,Serban],[Belitsky,GK]

$$\mathbb{O}_\ell \sim \det_{1\le n,m<\infty}(\delta_{nm}-K_{nm}(g))$$

Similar det-representation has been found for other observables in SYM theories [Many people]

- ✔ V.e.v. of half-BPS circular Wilson loop in $\mathcal{N}=4$ SYM
- ✔ Correlation function of infinitely heavy half-BPS operators (= octagon)
- ✔ Flux tube correlators (cusp anom. dim., scattering amplitudes)
- ✔ Free energy and correlation functions in $\mathcal{N}=2$ SYM

Truncated Bessel matrix/kernel

Across different observables, the primary object of investigation is

$$D_\ell(g) = \det\left(\delta_{nm} - K_{nm}(g)\right)\Big|_{1\le n,m<\infty}$$

The matrix elements admit an integral representation in terms of Bessel functions

$$K_{nm}(g) = \int_0^\infty dx\, \psi_n(x)\chi\Big(\frac{\sqrt{x}}{2g}\Big)\psi_m(x)$$

$$\psi_n(x) = \sqrt{2n+\ell-1}\,\frac{J_{2n+\ell-1}(\sqrt{x})}{\sqrt{x}}$$

The determinant depends on the coupling g, the bridge length ℓ and the *symbol* of the matrix $\chi(x)$

Special cases of the symbol

- ✔ $\chi(x) = 1$: the Bessel matrix simplifies as $K_{nm} = \delta_{nm}$, the determinant vanishes $D(g) = 0$
- ✔ $\chi(x) = \theta(1-x)$: $D_\ell(g)$ coincides with the Tracy-Widom distribution
- ✔ Various symbol functions in SYM theories, e.g.

$$\chi_{\text{flux tube}}(x) = 1 - \coth(x/2)$$

Flux tube correlators

The determinant depends on the symbol function $\chi(x)$ in a nontrivial way

For generic $\chi(x)$, we can expand $D_\ell(g)$ at small and large g, and interpolate between them.

For the ‘flux tube’, $D_\ell(g)$ can be computed exactly for the first few ℓ’s

$$D_{\ell=0}(g) = \left[\frac{2\pi g\cosh^3(2\pi g)}{\sinh(2\pi g)}\right]^{1/8}$$

$$D_{\ell=1}(g) = \left[\frac{\sinh^3(2\pi g)}{(2\pi g)^3\cosh(2\pi g)}\right]^{1/8}$$

$$D_{\ell=2}(g) = \frac{\log(\cosh(2\pi g))}{2(\pi g)^2}\left[\frac{2\pi g\cosh^3(2\pi g)}{\sinh(2\pi g)}\right]^{1/8}$$

Is it possible to obtain exact expressions for higher ℓ?

Can D_ℓ be expressed using ‘elementary’ functions?

Otherwise, what is the suitable space of ‘special’ functions?

Master equation

For *arbitrary* symbol function $\chi(x)$, the determinant satisfies a differential-difference equation

$$g\partial_g \log\left(\frac{D_{\ell+1}}{D_{\ell-1}}\right) = 2\ell\left(\frac{D_\ell^2}{D_{\ell-1}D_{\ell+1}} - 1\right)$$

This equation emerged from the study of correlation functions in $\mathcal{N}=2$ SYM using integrability [Ferrando,Komatsu,Lefundes,Serban] and localization [GK,Testa], for the special choice $\chi(x) = -\sinh^{-2}(\frac{x}{2})$

Supplemented with the boundary condition at weak coupling

$$D_\ell(g) = 1 + O(g^{2(\ell+1)})$$

it allows for a recursive determination of the function $D_{\ell+n}(g)$ for arbitrary $n \geq 1$, e.g.

$$D_{\ell+1}(g) = 2\ell D_{\ell-1}(g)\int_0^1 dx\, x^{2\ell-1}\left(\frac{D_\ell(xg)}{D_{\ell-1}(xg)}\right)^2$$

The function $D_{\ell+n}(g)$ can be expressed in terms of $D_{\ell-1}(g)$ and $D_\ell(g)$

Due to nonlinearity on the right-hand side, there is little hope of finding a closed-form expression,

unless...

Some simplifications

Examine separately even and odd n and introduce the ratios

$$\frac{D_{\ell+2n-1}}{D_{\ell-1}} = \mathcal{N}_{2n} \frac{d_{2n}(g)}{g^{2n(n+\ell-1)}} \,, \qquad \frac{D_{\ell+2n}}{D_{\ell}} = \mathcal{N}_{2n+1} \frac{d_{2n+1}(g)}{g^{2n(n+\ell)}} \,,$$

where $\mathcal{N}_n$ is the normalization factor

The functions $d_n(g)$ satisfy the equations

$$d_0 = d_1 = 1$$

$$d_{2n} d'_{2n+2} - d_{2n+2} d'_{2n} = d^2_{2n+1} \, (\log f_+)'$$

$$d_{2n-1} d'_{2n+1} - d_{2n+1} d'_{2n-1} = d^2_{2n} \, (\log f_-)'$$

where $n \geq 0$ and prime denotes a derivative with respect of the coupling g

The functions $f_\pm(g)$ depend on the initial conditions, $D_{\ell-1}(g)$ and $D_\ell(g)$

$$(\log f_+(g))' = g^{2\ell-1} \frac{2D^2_\ell(g)}{D^2_{\ell-1}(g)} \,, \qquad (\log f_-(g))' = g^{1-2\ell} \frac{2D^2_{\ell-1}(g)}{D^2_\ell(g)}$$

First attempt

Start with $d_2(g)$

$$d_2(g) = \int_0^g dg_1 (\log f_+(g_1))' = \int_0^g d\log f_+(g_1)$$

Continue with $d_3(g)$

$$\begin{aligned} d_3(g) &= \int_0^g d\log f_-(g_1)\, d_2^2(g_1) \\ &= 2\int_0^g d\log f_-(g_1) \int_0^{g_1} d\log f_+(g_2) \int_0^{g_2} d\log f_+(g_3) \end{aligned}$$

Examine $d_4(g)$

$$d_4(g) = d_2(g) \int_0^g dg_1 \frac{d_3^2(g_1)}{d_2^2(g_1)} (\log f_+(g_1))' = d_2(g) \int_0^g dg_1 \frac{d_3^2(g_1)}{d_2^2(g_1)} d_2'(g_1)$$

Integrate by parts

$$d_4(g) = -d_3^2(g) + 2d_2(g) \int_0^g dg_1\, d_2(g_1) d_3(g_1) (\log f_-(g_1))' \,.$$

Nonlinearity disappeared! $d_4(g)$ can be expanded into a linear combination of iterated integrals

Iterated (Chen) integrals

The integrals $I_{\sigma_1\sigma_2\cdots\sigma_k}(g)$ depend on a sequence of signs $\sigma_i = \pm$

They are constructed by integrating the product of the derivatives $(\log f_\pm(g))'$

$$I_{\sigma_1\sigma_2\cdots\sigma_k}(g) = \int_0^g d\log f_{\sigma_1}(g_1) \int_0^{g_1} d\log f_{\sigma_2}(g_2) \cdots \int_0^{g_{k-1}} d\log f_{\sigma_k}(g_k)$$

Satisfy a recurrence relation

$$I_{\sigma_1\sigma_2\cdots\sigma_k}(g) = \int_0^g d\log f_{\sigma_1}(g_1)\, I_{\sigma_2\cdots\sigma_k}(g_1)$$

Their product can be expanded into a linear combination of integrals using the shuffle product

$$I_{\sigma_1\sigma_2\cdots\sigma_k}(g) I_{\sigma_{k+1}\cdots\sigma_{k+m}}(g) = \sum_{\sigma' \in (k,m)\ \text{shuffles}} I_{\sigma'_1\sigma'_2\cdots\sigma'_{k+m}}(g)$$

Example

$$I_+(g) I_{-++}(g) = 3I_{-+++}(g) + I_{+-++}(g)$$

The total number of '$+$' and '$-$' entries on both sides is preserved

The use of the iterated integrals

So far we obtained

$$d_2(g) = I_+(g)$$

$$d_3(g) = 2I_{-++}(g)$$

$$d_4(g) = 12I_{+--+++}(g) + 4I_{+-+-++}(g)$$

A general solution for d_n is given by linear combinations of $I_{\sigma_1\sigma_2\ldots}$ with integer positive coefficients

$$d_n(g) = \sum c_{\sigma_1\sigma_2\ldots\sigma_k} I_{\sigma_1\sigma_2\cdots\sigma_k}(g)$$

The sum runs over sequences $(\sigma_1\sigma_2\ldots\sigma_k)$ of length $k = n(n-1)/2$

The number of '$+$' and '$-$' entries depends on the parity of n

$$n = 2p: \qquad k_+ = p^2\,, \qquad k_- = p(p-1)\,,$$

$$n = 2p+1: \qquad k_+ = p(p+1)\,, \qquad k_- = p^2\,.$$

Solutions

The equations for $d_n(g)$ lead to an overdetermined linear system for the coefficients $c_{\sigma_1\sigma_2\ldots\sigma_k}$

The excess of equations over unknowns grows exponentially with n

The resulting expression for $d_5(g)$ is

$$d_5(g) = 16\Big(I_{-+-+-++-++} + I_{-++-+-+-++} + I_{-+-++-+-++}$$
$$+ 3I_{-+-+-+-+++} + 3I_{-+-++--+++} + 3I_{-++-+--+++} + 3I_{-++--++-++}$$
$$+ 6I_{-+-+--++++} + 9I_{-++--+-+++} + 18I_{-++---++++}\Big)$$

The total number of terms equals $(k_+ + k_-)!/(k_+!k_-!) = 210$ but numerous $c-$coefficients vanish

For different n, the total number of coefficients and the number of non-zero coefficients are

n	2	3	4	5	6	7
Total	1	3	15	210	5005	293930
Non-zero	1	1	2	10	120	3276

These numbers admit a simple interpretation in terms of path counting on a 2d square lattice lattice

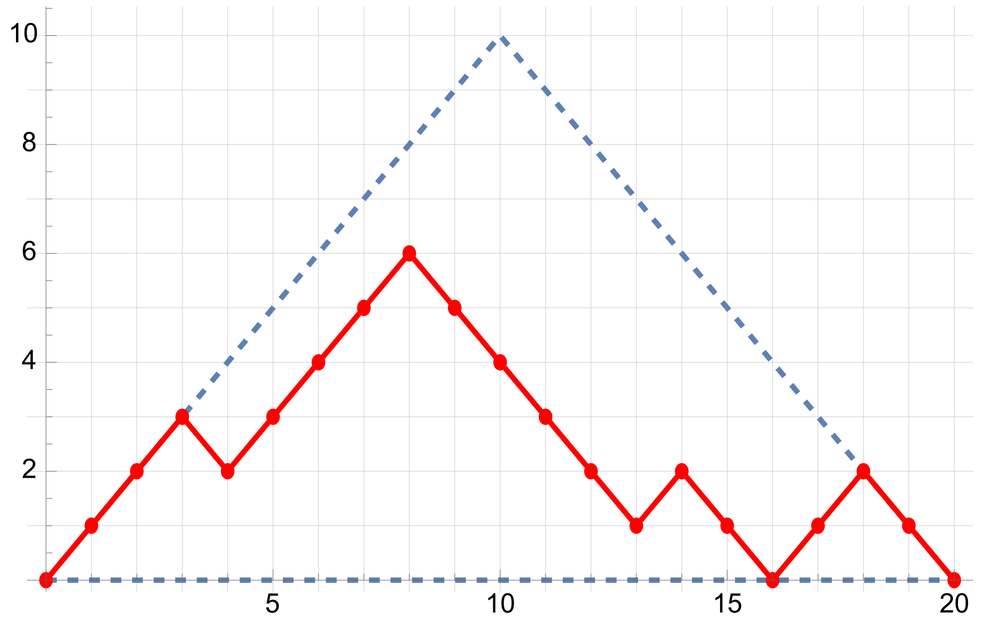
Solution for $n = 6$

$d_6(g) = 64$

$$\times \Big(30I_{+--+-+-+--+++++} + 18I_{+--+-+-+-+-++++} + 9I_{+--+-+-+-++-+++} + 3I_{+--+-+-+-+++-++}$$
$$+ 24I_{+--+-+-++--++++} + 12I_{+--+-+-++-+-+++} + 4I_{+--+-+-++-++-++} + 9I_{+--+-+-+++--+++}$$
$$+ 3I_{+--+-+-+++-+-++} + 90I_{+--+-++---+++++} + 54I_{+--+-++--+-++++} + 27I_{+--+-++--++-+++}$$
$$+ 9I_{+--+-++--+++-++} + 42I_{+--+-++-+--++++} + 21I_{+--+-++-+-+-+++} + 7I_{+--+-++-+-++-++}$$
$$+ 12I_{+--+-++-++--+++} + 4I_{+--+-++-++-+-++} + 54I_{+--+-+++---++++} + 27I_{+--+-+++--+-+++}$$
$$+ 9I_{+--+-+++--++-++} + 9I_{+--+-+++-+--+++} + 3I_{+--+-+++-+-+-++} + 90I_{+--++--+--+++++}$$
$$+ 54I_{+--++--+-+-++++} + 27I_{+--++--+-++-+++} + 9I_{+--++--+-+++-++} + 72I_{+--++--++--++++}$$
$$+ 36I_{+--++--++-+-+++} + 12I_{+--++--++-++-++} + 27I_{+--++--+++--+++} + 9I_{+--++--+++-+-++}$$
$$+ 270I_{+--++-+---+++++} + 162I_{+--++-+--+-++++} + 81I_{+--++-+--++-+++} + 27I_{+--++-+--+++-++}$$
$$+ 96I_{+--++-+-+--++++} + 48I_{+--++-+-+-+-+++} + 16I_{+--++-+-+-++-++} + 21I_{+--++-+-++--+++}$$
$$+ 7I_{+--++-+-++-+-++} + 72I_{+--++-++---++++} + 36I_{+--++-++--+-+++} + 12I_{+--++-++--++-++}$$
$$+ 12I_{+--++-++-+--+++} + 4I_{+--++-++-+-+-++} + 540I_{+--+++----+++++} + 324I_{+--+++---+-++++}$$
$$+ 162I_{+--+++---++-+++} + 54I_{+--+++---+++-++} + 162I_{+--+++--+--++++} + 81I_{+--+++--+-+-+++}$$
$$+ 27I_{+--+++--+-++-++} + 27I_{+--+++--++--+++} + 9I_{+--+++--++-+-++} + 54I_{+--+++-+---++++}$$
$$+ 27I_{+--+++-+--+-+++} + 9I_{+--+++-+--++-++} + 9I_{+--+++-+-+--+++} + 3I_{+--+++-+-+-+-++}$$
$$+ 30I_{+-+--+-+--+++++} + 18I_{+-+--+-+-+-++++} + 9I_{+-+--+-+-++-+++} + 3I_{+-+--+-+-+++-++}$$
$$+ 24I_{+-+--+-++--++++} + 12I_{+-+--+-++-+-+++} + 4I_{+-+--+-++-++-++} + 9I_{+-+--+-+++--+++}$$
$$+ 3I_{+-+--+-+++-+-++} + 90I_{+-+--++---+++++} + 54I_{+-+--++--+-++++} + 27I_{+-+--++--++-+++}$$
$$+ 9I_{+-+--++--+++-++} + 42I_{+-+--++-+--++++} + 21I_{+-+--++-+-+-+++} + 7I_{+-+--++-+-++-++}$$
$$+ 12I_{+-+--++-++--+++} + 4I_{+-+--++-++-+-++} + 54I_{+-+--+++---++++} + 27I_{+-+--+++--+-+++}$$
$$+ 9I_{+-+--+++--++-++} + 9I_{+-+--+++-+--+++} + 3I_{+-+--+++-+-+-++} + 30I_{+-+-+--+--+++++}$$
$$+ 18I_{+-+-+--+-+-++++} + 9I_{+-+-+--+-++-+++} + 3I_{+-+-+--+-+++-++} + 24I_{+-+-+--++--++++}$$
$$+ 12I_{+-+-+--++-+-+++} + 4I_{+-+-+--++-++-++} + 9I_{+-+-+--+++--+++} + 3I_{+-+-+--+++-+-++}$$
$$+ 90I + 54I + 27I + 9I$$

Dyck path

A particular example of a lattice path



Starts at the origin $(0, 0)$, ends on the $x-$axis, and never dips below it

Assign signs '$+$' and '$-$' to the up- and down-steps, respectively

A path can be represented as a sequence $(\sigma_1\sigma_2\dots\sigma_k)$, where $\sigma_i = \pm$ corresponds to the ith step:

$$(+++-++++-----+--++--)$$

Relation to lattice paths

General solution

$$d_n(g) = \sum c_{\sigma_1\sigma_2\ldots\sigma_k} I_{\sigma_1\sigma_2\cdots\sigma_k}(g)$$

The iterated integrals $I_{\sigma_1\sigma_2\cdots\sigma_k}(g)$ depend on the initial conditions $D_{\ell-1}(g)$ and $D_\ell(g)$

The coefficients $c_{\sigma_1\sigma_2\ldots\sigma_k}$ take positive integers and *universal*: they depend only on nonnegative integer n

Q: Is it possible to construct $d_n(g)$ without performing any explicit calculations?

Main idea: interpret each term in the sum as corresponding to a path on the square lattice, uniquely determined by the sequence $(\sigma_1\sigma_2\cdots\sigma_k)$

> *The function $d_n(g)$ is a partition function (or generating function in enumerating combinatorics) of an ensemble of lattice paths confined to a nontrivial domain*

Warm up example

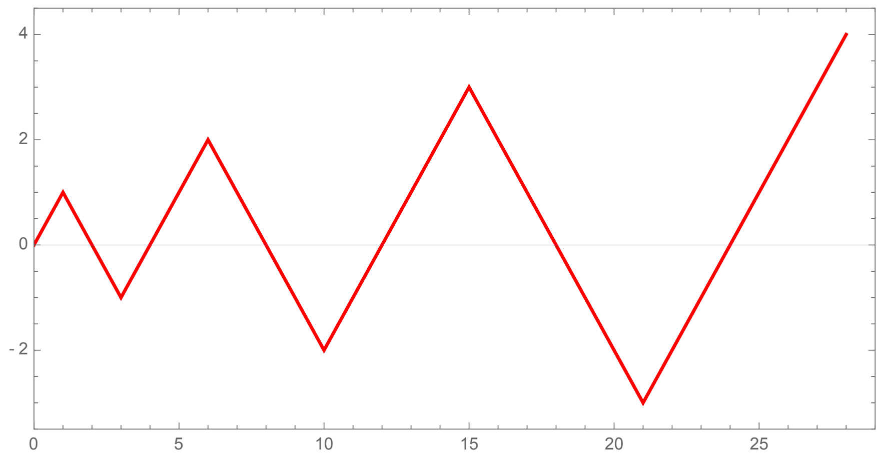
Examples of paths for d_n at $n = 8$:

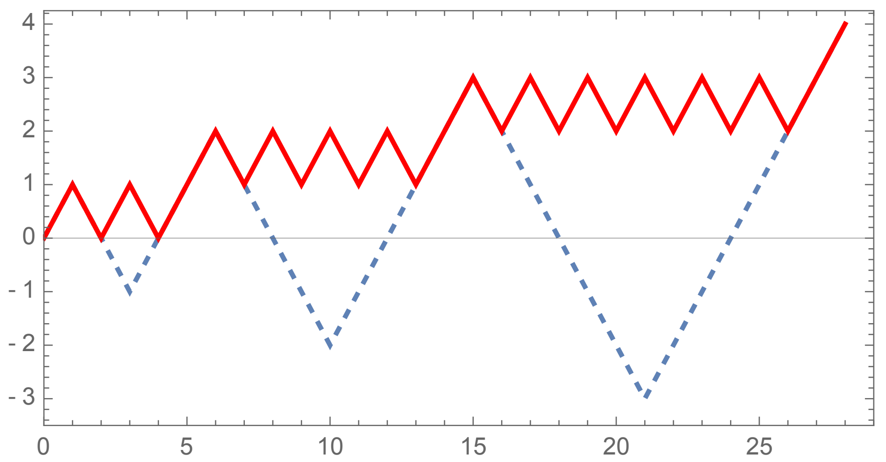
$$c_{+--+++----+++++------+++++++}$$

$$c_{+-+-++-+-+-+-++-+-+-+-+-+-++}$$

Total length $k = n(n-1)/2 = 28$, number of '$+$' and '$-$' is $k_+ = 16$ and $k_- = 12$

The corresponding paths are





For all coefficients in d_8, the paths begin at the origin $(0,0)$ and terminate at the same point

$$p_n = (n(n-1)/2, (-1)^n \lfloor n/2 \rfloor)$$

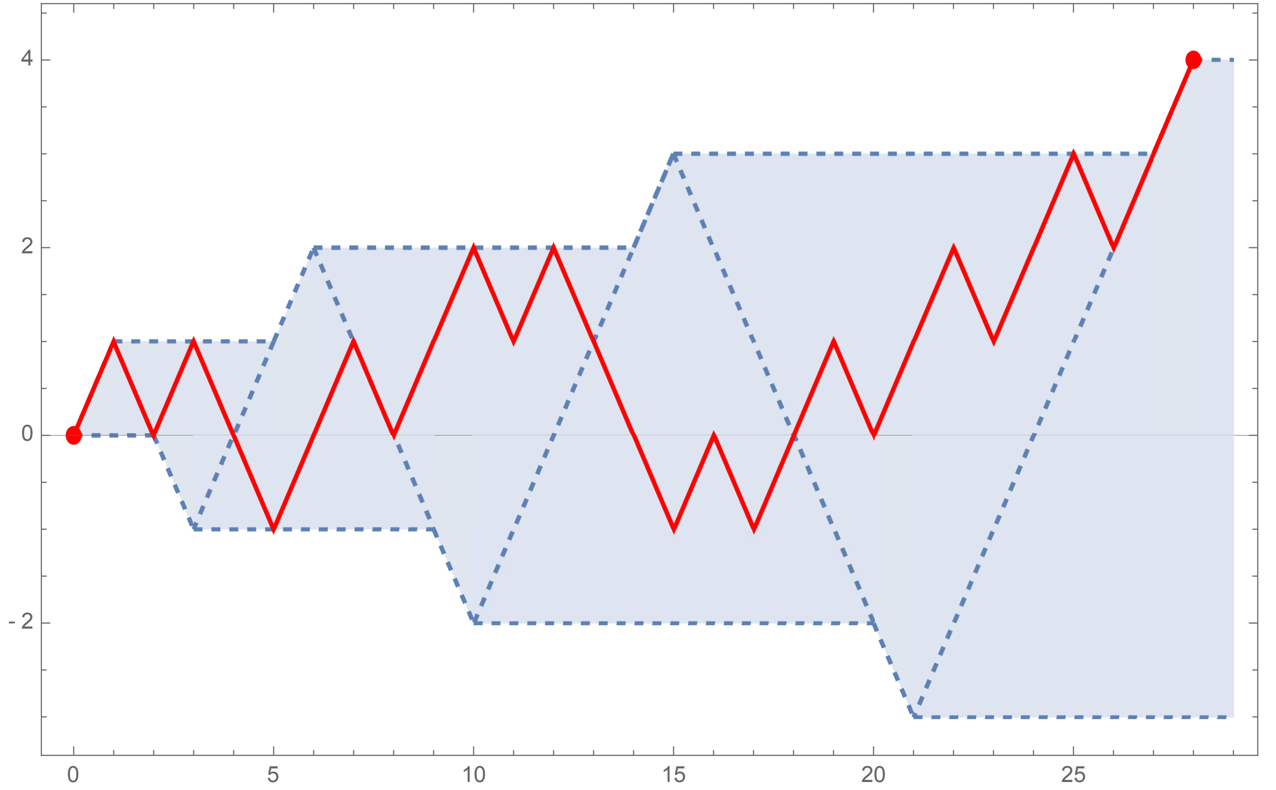
regardless of the order in which the '$+$' and '$-$' signs appear in the sequence.

Admissible paths

The total number of possible coefficients is $(k_+ + k_-)!/(k_+!k_-!)$

The number of nonzero coefficients is significantly smaller:

All admissible paths must terminate at the point p_n and remain entirely within the envelope defined by the shaded region. Paths that violate this rule do not contribute.



Application to the flux tube correlators

Recall that D_ℓ (for $\ell = 0, 1, 2$) were given by elementary functions (notation $z = e^{-4\pi g}$)

$$D_0(g) = \left[\frac{(1+z)^3 \log(1/z)}{8(1-z)z}\right]^{1/8}, \qquad D_1(g) = \left[\frac{2(1-z)^3}{z(1+z)\log^3(1/z)}\right]^{1/8}$$

The iterated integrals are built out of

$$(\log f_+(g))' = \frac{1}{\pi}\frac{1-z}{1+z}, \qquad (\log f_-(g))' = 4\pi\frac{1+z}{1-z}$$

They admit a *d-log* representation, e.g.

$$I_{-++}(g) = \frac{1}{2\pi^4}\int_1^z d\log\left(\frac{1-z_1}{2\sqrt{z_1}}\right)\int_1^{z_1} d\log\left(\frac{1+z_2}{2\sqrt{z_2}}\right)\int_1^{z_2} d\log\left(\frac{1+z_3}{2\sqrt{z_3}}\right)$$

Can be evaluated in terms of harmonic polylogarithms (HPL) of weight 3.

Finally,

$$D_3(g) = 4 I_{-++}(g) D_1(g)/g^4$$

$D_n(g) \sim$ [Multi-linear combination of HPL's of weight $n(n-1)/2$]

Conclusions

- ✔ A broad class of observables in four-dimensional $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal Yang-Mills theories are computable in the planar limit at finite 't Hooft coupling
- ✔ These observables admit a representation as Fredholm determinants of integrable Bessel kernels and satisfy a universal differential-difference equation
- ✔ This powerful equation allows for the recursive determination of its solutions in terms of iterated Chen integrals
- ✔ The observables admit a natural interpretation in terms of enumerative combinatorics: they can be identified with the partition function (or generating function) of an ensemble of lattice paths

Take-home message