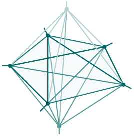
# Non-Local Symmetries of Planar Feynman Integrals

Florian Loebbert



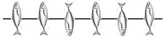
Bethe Center For Theoretical Physics
University of Bonn

based on
arxiv:2410.11936 with H.Mathur
arxiv:2505.05550 with L.Rüenaufer and S.Stawinski
and work in progress with G.Ferrando, A.Mierau and S.Stawinski

**Fishnets** 2024 **2025** 2026 2027 . . .
**Southampton**

# Motivation

**Simple integrable fishnet QFTs** arise from double-scaling limits of deformed $\mathcal{N} = 4$ SYM theory [Kazakov Gürdoğan 2016]

- AdS/CFT-type integrability in a simpler framework
- many interesting QFT results  [by Participants of Fishnets 24 & 25]
- fishnet correlators identified with Feynman integrals [cf. Deliang's Talk]

**Computation of Feynman integrals** poses hard problems

- bottle neck for phenomenological predictions (collider experiments, gravitational waves)
- interesting mathematical problems (function classes relevant in other areas of physics)
- many recent developments, see e.g. [review 2203.07088 by Bourjaily, Broedel, Chaubey, Duhr, Frellesvig, Hidding, Marzucca, McLeod Spradlin, Tancredi, Vergu, Volk, Volovich, von Hippel, Weinzierl, Wilhelm, Zhang]

# Brief Review

# Integrability and Yangian Symmetry

- The Yangian is an infinite dimensional extension of a Lie algebra $\mathfrak{g}$.
- It underlies rational quantum integrable models (e.g. $\mathrm{AdS}_5/\mathrm{CFT}_4$).

Yangian algebra $Y[\mathfrak{g}]$ (first realization): [Drinfel'd 1985]

Level 0 : $$\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}^a_k \in \mathfrak{g}, \qquad [\mathrm{J}^a, \mathrm{J}^b] = f^{ab}{}_c \mathrm{J}^c$$

Level 1 : $$\widehat{\mathrm{J}}^a|_{1,n} = f^a{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}^c_j \, \mathrm{J}^b_k, \qquad [\mathrm{J}^a, \widehat{\mathrm{J}}^b] = f^{ab}{}_c \widehat{\mathrm{J}}^c$$

Serre relations: $$[\widehat{\mathrm{J}}_a, [\widehat{\mathrm{J}}_b, \mathrm{J}_c]] - [\mathrm{J}_a, [\widehat{\mathrm{J}}_b, \widehat{\mathrm{J}}_c]] = \mathcal{O}(\mathrm{J}^3).$$

- $Y[\mathfrak{psu}(2,2|4)]$ realized in $\mathcal{N}=4$ SYM theory
- $Y[\mathfrak{so}(2,D)]$ annihilates fishnet correlators (alias Feynman integrals)

# Conformal Yangian and Feynman Integrals

**Differential operator representation:**

$$\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}^a_k \qquad \text{with} \qquad \mathrm{J}^a \in \begin{cases} \mathrm{D} = -ix_\mu \partial^\mu - i\Delta, \\ \mathrm{L}_{\mu\nu} = ix_\mu \partial_\nu - ix_\nu \partial_\mu, \\ \mathrm{P}_\mu = -i\partial_\mu, \\ \mathrm{K}_\mu = ix^2 \partial_\mu - 2ix_\mu x^\nu \partial_\nu - 2i\Delta x_\mu. \end{cases}$$

**Integrability for Feynman integrals:** Conformal Yangian symmetry

level zero: $$\underbrace{\{\mathrm{P}^\mu, \mathrm{L}_{\mu\nu}, \mathrm{D}, \mathrm{K}^\mu\}}_{\text{conformal generators}} I_n = 0,$$

level one: $$\widehat{\mathrm{P}}^\mu I_n = 0.$$

**Non-local $\widehat{\mathbf{P}}^{\boldsymbol{\mu}}$** pushes the problem across the boundary to integrability.
(dual plus conf. symmetry, cf. [Drummond, Henn Korchemsky, Sokatchev '08] [Drummond Henn, Plefka '09] [FL, Miczajka Müller, Münkler '20])

# Conformal Yangian and Feynman Integrals

**Differential operator representation:**

$$\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}_k^a \qquad \text{with} \qquad \mathrm{J}^a \in \begin{cases} \mathrm{D} = -i x_\mu \partial^\mu - i\Delta, \\ \mathrm{L}_{\mu\nu} = i x_\mu \partial_\nu - i x_\nu \partial_\mu, \\ \mathrm{P}_\mu = -i\partial_\mu, \\ \mathrm{K}_\mu = i x^2 \partial_\mu - 2i x_\mu x^\nu \partial_\nu - 2i\Delta x_\mu. \end{cases}$$

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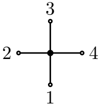
This talk: How general/useful are $\widehat{\mathrm{P}}$ symmetries?

# Example: Cross from Yangian Symmetry

Level-zero symmetry implies conformal variables $z, \bar{z}$:

$I_4 =$  $= \frac{1}{x_{13}^2 x_{24}^2} \phi(z, \bar{z})$



Level-one $\widehat{\mathrm{P}}$ differential equations:

$$[D_j(z) - D_j(\bar{z})]\phi = 0 \quad \text{with} \quad \begin{cases} D_1(z) = z(z-1)^2 \partial_z^2 + (3z-1)(z-1)\partial_z + z, \\ D_2(z) = z^2(z-1)\partial_z^2 + (3z-2)z\partial_z + z. \end{cases}$$

Four solutions $f_j(z, \bar{z})/(z - \bar{z})$ :

$$\begin{aligned} f_1 &= 1, \\ f_2 &= \log(\bar{z}) - \log(z), \\ f_3 &= \log(1-\bar{z}) - \log(1-z), \\ f_4 &= 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log \tfrac{1-z}{1-\bar{z}} \log(\bar{z}z). \end{aligned}$$

Permutations single out Bloch-Wigner dilogarithm. [Usyukina Davydychev '93] [FL, Müller Münkler '19]

# Yangian Symmetry and Lasso Method

**Define Yangian algebra** via RTT-relations for monodromy

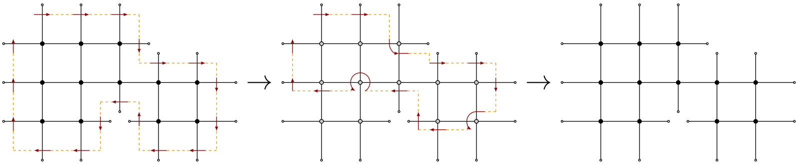
$T(u) =$  :

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$$

**Yangian invariance** means eigenvalue equation for $n$-point invariant $I_n$:

$$T(u)I_n = f(u)I_n \cdot \mathbb{1}, \qquad T(u) = f(u)\big(\,\mathbb{1} + \tfrac{1}{u}\mathrm{J} + \tfrac{1}{u^2}\widehat{\mathrm{J}} + \ldots .$$

**Proof of symmetry** by *lasso method*© [Chicherin, Kazakov, FL Müller, Zhong 2018] [Kazakov, Mishnyakov Levkovich-Maslyuk 2023]



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**Proof of symmetry** by *lasso method*© [Chicherin, Kazakov, FL Müller, Zhong 2017] [Kazakov, Mishnyakov Levkovich-Maslyuk 2023]

Disadvantages of this formulation:

- Graph-by-graph proof
- No generic ($D$-dimensional) R-matrix known
- No separation of Lie symmetry and integrability
- No inclusion of massive propagators yet

# Tree Graphs and Feyn-Structure of Yangian Symmetry

# The Level-one Momentum $\widehat{\mathrm{P}}^{\mu}$

Explicit form of level-one momentum operator:

$$\begin{aligned}\widehat{\mathrm{P}}^{\mu} &= \tfrac{1}{2} f^{\mathrm{P}^{\mu}}{}_{bc} \sum_{j=1}^{n} \sum_{k=j+1}^{n} \mathrm{J}_j^c \mathrm{J}_k^b + \sum_{j=1}^{n} s_j \mathrm{P}_j^{\mu} \\ &= \frac{i}{2} \sum_{j=1}^{n} \sum_{k=j+1}^{n} \left( \mathrm{P}_j^{\mu} \mathrm{D}_k + \mathrm{P}_{j\nu} \mathrm{L}_k^{\mu\nu} - \mathrm{D}_j \mathrm{P}_k^{\mu} - \mathrm{L}_j^{\mu\nu} \mathrm{P}_{k\nu} \right) + \sum_{j=1}^{n} s_j \mathrm{P}_j^{\mu}.\end{aligned}$$

with conformal Lie algebra generators:

$$\mathrm{P}_j^{\mu} = -i \partial_{x_j}^{\mu}, \qquad \mathrm{L}_j^{\mu\nu} = i(x_j^{\mu} \partial_{x_j}^{\nu} - x_j^{\nu} \partial_{x_j}^{\mu}), \qquad \mathrm{D}_j = -i(x_{j\mu} \partial_{x_j}^{\mu} + \Delta_j),$$

and evaluation parameters $s_j$.

# Two-Point Symmetries

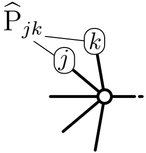
**$\widehat{\mathrm{P}}$ on two propagators:**

$$\widehat{\mathrm{P}}^{\mu}_{jk}\,\frac{1}{(x_{j0}^2)^{a_j}(x_{k0}^2)^{a_k}} = 0, \qquad\qquad x_{j0}^{\mu} = x_j^{\mu} - x_0^{\mu}$$

for

$$\widehat{\mathrm{P}}^{\mu}_{jk} := \frac{i}{2}\left(\mathrm{P}^{\mu}_j \mathrm{D}_k + \mathrm{P}_{j\nu}\mathrm{L}^{\mu\nu}_k - i a_k \mathrm{P}^{\mu}_j - (j \leftrightarrow k)\right),$$

**Two-Point Symmetries of general graphs:** [FL, Miczajka Müller, Münkler '20]



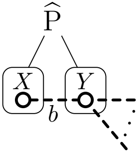
# Two-Vertex Symmetries

**Generalization:** [FL, Mathur 2024]

$$\widehat{\mathrm{P}}^{\mu}_{jk} \frac{1}{(x^2_{jX})^{a_j}(x^2_{kY})^{a_k}} = 2ia_j a_k \frac{T^{\alpha\beta\mu\nu} x_{XY\nu}\, x_{jX\alpha}\, x_{kY\beta}}{(x^2_{jX})^{a_j+1}(x^2_{kY})^{a_k+1}},$$

with $T^{\alpha\beta\mu\nu} := \eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu}$, allows to show

**End-Vertex Symmetries:**



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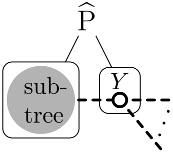

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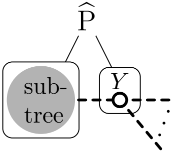
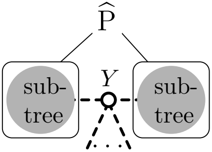


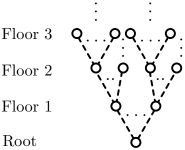

**Bridge-Vertex Symmetries:**



# Yangian Feyn-Structure

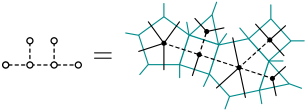
**Full $\widehat{\mathrm{P}}$ symmetry of all (position-space) trees** is rigorously proven by decomposing the generator into the above sub-symmetries. [FL, Mathur 2024]

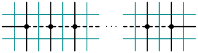
$$\widehat{\mathrm{P}}^{\mu} \simeq \sum_{p=1}^{\ell} \sum_{j \in V_{X_p}} \sum_{\substack{k \in V_{X_p} \\ k > j}} \widehat{\mathrm{P}}^{\mu}_{jk} + \sum_{p=1}^{\ell} \sum_{q=p+1}^{\ell} \widehat{\mathrm{P}}^{\mu}_{X_p X_q}$$

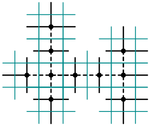



**No constraints on the propagator powers required!**

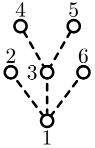
Includes for instance train-track and train-track network Feynman graphs relating to Calabi–Yau geometries [Bourjaily, He, McLeod von Hippel, Wilhelm 2018] [Duhr, Klemm, FL Nega, Porkert '22-24] [McLeod von Hippel 2023]:







# Six-Loop Example




**Partial $\widehat{\mathrm{P}}$ symmetries given by**

End-vertex symmetries:

$$\widehat{\mathrm{P}}^{\mu}_{X_1X_2}, \quad \widehat{\mathrm{P}}^{\mu}_{X_1X_6}, \quad ,\widehat{\mathrm{P}}^{\mu}_{X_3X_4}, \quad ,\widehat{\mathrm{P}}^{\mu}_{X_3X_5} \quad \sum_{q=3}^{5} \widehat{\mathrm{P}}^{\mu}_{X_1X_q}$$

Bridge-vertex symmetries:

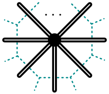
$$\widehat{\mathrm{P}}^{\mu}_{X_2X_6}, \quad \sum_{q=3}^{5} \widehat{\mathrm{P}}^{\mu}_{X_2X_q}, \quad \sum_{q=3}^{5} \widehat{\mathrm{P}}^{\mu}_{X_qX_6}, \quad \widehat{\mathrm{P}}^{\mu}_{X_4X_5}$$

**Full Yangian level-one** $\widehat{\mathrm{P}}$ from combining these.

# Massive Extension

Introduce massive propagators for boundary legs:

$$\frac{1}{x_{jk}^2} \to \frac{1}{x_{jk}^2 + (m_j - m_k)^2} \qquad m_{\text{internal}} = 0,$$

e.g. at one loop: $\quad I_n^{m_j} = \int \frac{d^D x_0}{\prod_{j=1}^n (x_{j0}^2 + m_j^2)^{a_j}} =$ 

Massive extension of conformal Lie algebra [Alday, Henn Plefka, Schuster '09]

$$\mathrm{P}_j^\mu = -i\partial_j^\mu, \qquad \mathrm{L}_j^{\mu\nu} = i(x_j^\mu \partial_j^\nu - x_j^\nu \partial_j^\mu),$$
$$\mathbb{D}_j = -i(x_{j\mu}\partial_j^\mu + m_j \partial_{m_j} + \Delta_j),$$
$$\mathbb{K}_j^\mu = -i(2x_j^\mu(x_j^\nu \partial_{j\nu} + m_j \partial_{m_j}) - (x_j^2 + m_j^2)\partial_j^\mu + 2\Delta_j x_j^\mu),$$

Above proof of non-local $\widehat{\mathrm{P}}$ symmetries generalizes with: [FL, Mathur 2024]

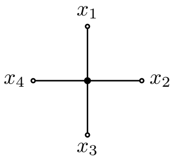
$$\widehat{\mathbb{P}}_{jk}^\mu \frac{1}{(x_{jX}^2 + m_j^2)^{a_j}(x_{kY}^2 + m_k^2)^{a_k}} = \frac{2i a_j a_k T^{\alpha\beta\mu\nu} x_{XY\nu}\, x_{jX\alpha}\, x_{kY\beta}}{(x_{jX}^2 + m_j^2)^{a_j+1}(x_{kY}^2 + m_k^2)^{a_k+1}}$$

# Trees to Loops?

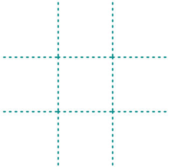
We have proven $\widehat{\mathrm{P}}$ (sub-)symmetries of **all position-space tree graphs**

**How about position-space loops?** $\rightarrow$ use dual picture:

$$x_j^\mu \qquad\qquad p_j^\mu = x_j^\mu - x_{j+1}^\mu$$




$\rightarrow$



$$\int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} \qquad\qquad \int \frac{\mathrm{d}^4 \ell}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$

# $\widehat{\mathrm{P}}$ Symmetry for all Planar Graphs

# $\widehat{\mathrm{P}}^\mu$ vs $\bar{\mathrm{K}}^\mu$

Go to dual momentum space defined via: $p_j = x_j - x_{j+1}$ (planar graphs)

On invariants $I_n$ under $\mathrm{P}^\mu, \mathrm{L}^\mu{}_\nu, \mathrm{D}$ one finds [FL, Miczajka Müller, Münkler '20] [cf. Drummond Henn, Plefka '09]

$$(x\text{-space}) \qquad \widehat{\mathrm{P}}^\mu \simeq \bar{\mathrm{K}}^\mu \qquad (p\text{-space})$$

with momentum space conformal generator (or massive generalization)

$$\bar{\mathrm{K}}^\mu = \sum_{i=1}^{n-1} \left( p_i^\mu \partial_{p_i}^2 - 2 p_i^\nu \partial_{p_i,\nu} \partial_{p_i}^\mu - 2 \bar{\Delta}_i \partial_{p_i}^\mu \right),$$

where

$$\bar{\Delta}_i = \frac{1}{2}(\Delta_i + \Delta_{i+1} + 2s_i - 2s_{i+1}), \qquad i = 1, \ldots, n-1\,.$$

Easier to show $\bar{\mathrm{K}}^\mu$ invariance?

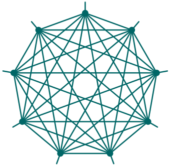
# Momentum Space Conformal Invariants

What are solutions to momentum space conformal constraints?

$$\bar{\mathrm{K}}^\mu(\bar{\Delta}_i) M_n^\delta(p_1, \ldots, p_n) = 0 \,.$$

**Answer:** Simplex and Mesh integrals (proved by induction) [Bzowski,McFadden Skenderis 2019-20]

$$M_n^\delta = \prod_{i<j} \int \frac{\mathrm{d}^D q_{ij}}{\pi^{D/2}} \frac{C_{ij}}{q_{ij}^{2(\alpha_{ij}+\frac{D}{2})}} \prod_{k=1}^{n} \delta\Big(p_k + \sum_{i=1}^{n} q_{ik}\Big),$$



where

$$\bar{\Delta}_i = D + \sum_{j=1}^{n} \alpha_{ij}, \qquad C_{ij} = \frac{4^{\alpha_{ij}}}{\Gamma(-\alpha_{ij})} \Gamma\left(\tfrac{D}{2} + \alpha_{ij}\right)$$

Strip off delta function:

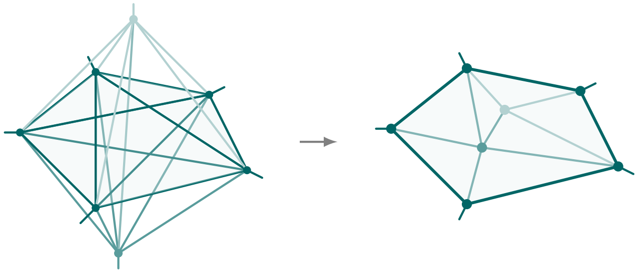
$$M_n^\delta(p_1, \ldots, p_n) = \delta\left(\sum_{i=1}^{n} p_i\right) M_n(p_1, \ldots, p_{n-1}) \,.$$

Generic propagator powers $\alpha_{ij}$!

# Idea

Prove $\widehat{\mathrm{P}}$-invariance of position-space Feynman integrals $I_n$ by [FL, Rüenaufer Stawinski '25]

- planarizing mesh integrals and
- going to dual position space $(p_j \to x_j - x_{j+1})$



# Tree Graphs

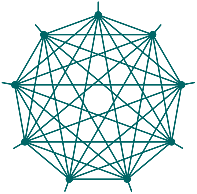
Remove propagators (and integrals) using identity

$$\lim_{\alpha\to 0} \frac{1}{\Gamma(-\alpha)} \frac{1}{q^{2(D/2+\alpha)}} = \frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)} \delta^{(D)}(q)\,,$$

Any position-space tree can be obtained from a mesh:

Mesh

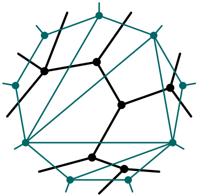
$$0 = \lim_{\alpha_{ij}\to 0} \bar{\mathrm{K}}^{\mu}$$



$M_n$

Tree Graph

$$= \widehat{\mathrm{P}}^{\mu}$$



$I_n$

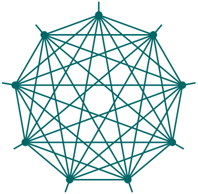
$\Rightarrow \widehat{\mathrm{P}}^{\mu}$ symmetry follows from $\bar{\mathrm{K}}^{\mu}$ symmetry (<u>no constraints at all</u>)

# Loop Graphs

Remove propagators and take soft limits

Mesh

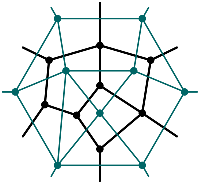
$$0 = \lim_{\substack{\alpha_{ij}\to 0\\ p_{1,2,3}\to 0}} \bar{\mathrm{K}}^\mu$$



$M_n$

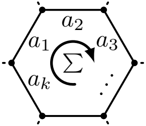
$$\overset{\bar{\Delta}_i=0}{=} \quad \widehat{\mathrm{P}}^\mu$$

Loop Graph



$I_n$

Soft limit $p_i \to 0$ only commutes with action of $\bar{\mathrm{K}}^\mu$ if $\bar{\Delta}_i = 0$:



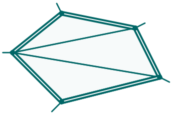

$$= \sum_{m=1}^{k} a_m = \frac{(k-2)D}{2}$$

[observed on examples in] [Kazakov, Mishnyakov Levkovich-Maslyuk '23] [Levkovich-Maslyuk Mishnyakov '24]

$\Rightarrow \widehat{\mathrm{P}}^\mu$ symmetry follows from $\bar{\mathrm{K}}^\mu$ symmetry for above constraints.

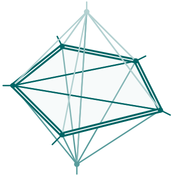
# Massive Meshes

[cf. Kostya's Talk]



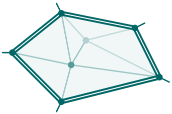
Step 1

Start with planar mesh with massive boundary, dual (tree) graph invariant under $\widehat{\mathbb{P}} \simeq \bar{\mathbb{K}}^\mu$ [FL '24 Mathur]



Step 2

Use induction of [Bzowski, McFadden Skenderis '20], to extend $\bar{\mathbb{K}}^\mu$ invariance to non-planar mesh.



Step 3

Planarize via soft and propagator-power limits, which preserve the $\widehat{\mathbb{P}}^\mu$ symmetry on constraints.

---

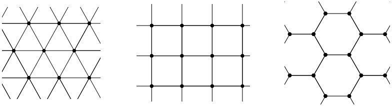
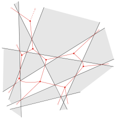
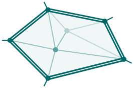
Massive generalization of momentum space generator [FL, Miczajka Müller, Münkler '20]

$$\widehat{\mathbb{P}} \quad \to \quad \bar{\mathbb{K}}_j^\mu = \bar{\mathrm{K}}_j^\mu - (m_j \partial_{m_j} + m_{j+1}\partial_{m_{j+1}})\partial_{p_j}^\mu \,.$$

Ordinary, not dual conformal symmetry!

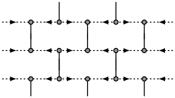
# Feynman Graphs and $\widehat{\mathrm{P}}$ Symmetry

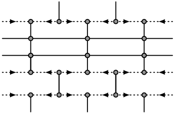
Above proof covers all previous (scalar) Yangian invariants and generalizes to graphs that are not invariant under conformal symmetry:

| **Fishnets** | **Looms** | **Massive Boundaries** |
|---|---|---|
|  |  |  |
| [Zamolodchikov 1980] [Chicherin, Kazakov FL, Müller, Zhong '17] | [Kazakov, Mishnyakov Levkovich-Maslyuk '23] | [FL, Miczajka Müller, Münkler '20] |

**Not (yet) Fermions**

e.g. brick wall graphs: [Chicherin, Kazakov FL, Müller, Zhong 2017] $\rightarrow$ [cf. Moritz' Talk]





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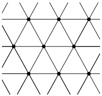
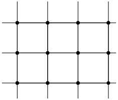
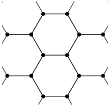
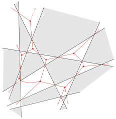
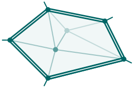
|    |  |  |
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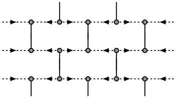


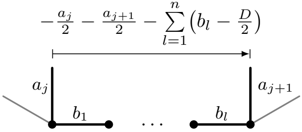
# Holographic Evaluation Parameters

Above proof yields evaluation parameters $\color{red}{s_j}$ in

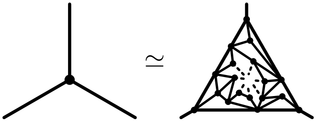
$$\widehat{\mathrm{P}}^{\mu} = \frac{i}{2}\sum_{j<k}\left(\mathrm{P}_j^{\mu}\mathrm{D}_k + \mathrm{P}_{j\nu}\mathrm{L}_k^{\mu\nu} - (j \leftrightarrow k)\right) + \sum_{j=1}^{n} \color{red}{s_j}\color{black}\mathrm{P}_j^{\mu}.$$

as observed previously [Chicherin, Kazakov, FL Müller, Zhong 2017] [FL, Miczajka Müller, Münkler 2020] [Kazakov, Mishnyakov Levkovich-Maslyuk 2023]:

$$s_{j+1} = s_j - \frac{a_j}{2} - \frac{a_{j+1}}{2} - \sum_l \left(b_l - \frac{D}{2}\right),$$




$\widehat{\mathrm{P}}$ equations only depend on boundary data of propagator powers.




$\widehat{\mathrm{P}}$ analogue of star-triangle relation.

# Feynman Integrals from Differential Equations

# Feynman Integrals and Geometry

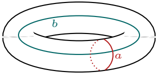
For fixed propagator powers, Feynman integrals connect to geometry

**Simplest example:** Conformal cross integral in 1D

$$I_4^{\mathrm{1D}} = \int \frac{\mathrm{d}x_0}{x_{10}^{2\frac{1}{4}} x_{20}^{2\frac{1}{4}} x_{30}^{2\frac{1}{4}} x_{40}^{2\frac{1}{4}}} \xrightarrow{\text{conf. transf.}} \int \frac{\mathrm{d}x}{\sqrt{x(x-1)(x-z)}} = \int \frac{dx}{y}$$

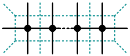
Natural geometry given by Legendre family of elliptic curves

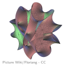
$$y^2 = x(x-1)(x-z)$$




$$I_4^{\mathrm{1D}} \simeq K(z) + K(1-z)$$

**Generalization:** Train tracks = Calabi–Yau $\ell$-folds [Duhr, Klemm, FL Nega, Porkert '22-24]



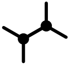


| Loops | 1 | 2 | 3 | ... |
|---|---|---|---|---|
| Geometry | Elliptic Curve | K3 surface | CY 3-fold | ... |

# Geometry and Graph Representation

**Example:** Two-loop three-point graph in 1D: [Duhr, Klemm, FL Nega, Porkert '24]

$$\int \frac{\mathrm{d}x_0 \mathrm{d}x_{\bar{0}}}{x_{10}^{\frac{2}{3}} x_{20}^{\frac{2}{3}} x_{0\bar{0}}^{\frac{2}{3}} x_{3\bar{0}}^{\frac{2}{3}} x_{4\bar{0}}^{\frac{2}{3}}} =$$



star-triangle $\simeq$

$$= \int \frac{\mathrm{d}x_0}{x_{10}^{\frac{2}{3}} x_{20}^{\frac{1}{3}} x_{30}^{\frac{2}{3}} x_{40}^{\frac{1}{3}}}$$

CY condition: integrand $\frac{1}{P_G^{(c-1)/c}}$ with polynomial of degree $n = \frac{2c}{c-1}$

<u>LHS:</u> Natural geometry is singular K3 with $c = n = 3$

$$y^3 = (x_1 - x_0)(x_2 - x_0)(x_0 - x_{\bar{0}})(x_3 - x_{\bar{0}})(x_4 - x_{\bar{0}})$$

<u>RHS:</u> Natural geometry is Picard-curve:

$$y^3 = (x_1 - x_0)^2 (x_2 - x_0)(x_3 - x_0)^2 (x_4 - x_0)$$

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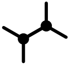

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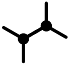

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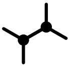

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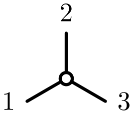
No unique geometry!

$\Rightarrow$ Motivates definition of integrals via differential equations

# Feynman Integrals and Hyper-Geometry

For generic propagator powers, every Feynman integral corresponds to a family of hypergeometric functions, see e.g. [Regge 1968][Kalmykov, Kniehl Ward, Yost 2008]

**Example: 1D triangle** with generic propagator powers ($\chi = \frac{x_{13}}{x_{12}}$)




$$\simeq \; A \, {}_2F_1 \left[ {2a_3, 2\sum_i a_i - 1 \atop 2(a_1 + a_3)} ; \chi \right] + B \, {}_2F_1 \left[ {2a_2, 1 - 2a_1 \atop 2(1 - a_1 - a_3)} ; \chi \right]$$

---

Prominent role played by Gelfand–Kapranov–Zelevinsky (GKZ) hypergeometric functions with defining set of differential equations

$\rightarrow$ GKZ maps to Yangian system for certain cases only [Levkovich-Maslyuk Mishnyakov 2024][Victor's talk]

Do the new $\widehat{\mathrm{P}}$ (sub-)symmetries define a Feynman integral?

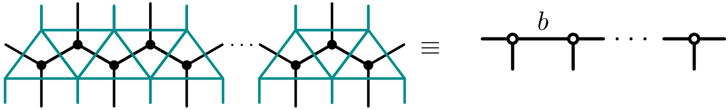
# Feynman Integrals in 1D

Consider Feynman integrals in one spacetime dimension

- with generic non-conformal propagator powers
- at position-space tree level

$\Rightarrow$ All $\widehat{\mathrm{P}}$ sub-symmetries apply without constraints on the powers.

**Triangle tracks**: Most general class of integrals with 'track' topology:

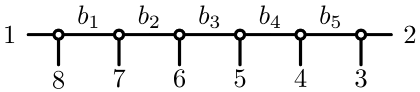



e.g. train tracks from shrinking internal propagators via $\lim_{b\to 0}$

Hypergeometry of $\ell$-loop triangle track fixed by: [Ferrando, FL Mierau, Stawinski in progress]

- 2 two-point symmetries
- $\ell - 2$ bridge-vertex symmetries

# Example: Six-Loop Triangle Track

$I_8 =$ 

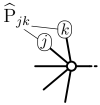
$= V_8\, \phi(\chi_1, \ldots, \chi_6)$

Scale invariant function depends on six ratios:

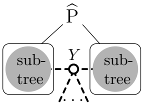
$$\chi_1 = \frac{x_{87}}{x_{81}}, \quad \chi_2 = \frac{x_{76}}{x_{78}}, \quad \chi_3 = \frac{x_{65}}{x_{67}}, \quad \chi_4 = \frac{x_{54}}{x_{56}}, \quad \chi_5 = \frac{x_{43}}{x_{45}}, \quad \chi_6 = \frac{x_{32}}{x_{34}}.$$

**$\widehat{\mathrm{P}}$ differential (sub-)symmetries:**

- 2 two-point symmetries




- 4 bridge-vertex symmetries



# Example: Six-Loop Triangle Track

$\widehat{\mathrm{P}}$ symmetries induce 6 recurrence equations for coefficient of the series:

$$\phi = \sum_{k,l,m,n,p,q} f_{k,l,m,n,p,q} \;\; \chi_1^k \chi_2^l \chi_3^m \chi_4^n \chi_5^p \chi_6^q$$

$\widehat{\mathrm{P}}$ constraints:

$$\begin{aligned}
&(-2a_8+2k-1)(4a_1+2a_8+2k-3)\, f_{k-1,l,m,n,p,q} = 2(2a_8+2k-1)(-a_7-a_8-2b_1+k-l+1)\, f_{k,l,m,n,p,q},\\
&(-2a_7+2l-1)(a_7+a_8+2b_1-k+l-2)\, f_{k,l-1,m,n,p,q} = -(2a_7+2l-1)(a_6+a_7+2b_2-l+m-1)\, f_{k,l,m,n,p,q},\\
&(2a_6+2m-1)(-a_5-a_6-2b_3+m-n+1)\, f_{k,l,m,n,p,q} = (-2a_6+2m-1)(a_6+a_7+2b_2-l+m-2)\, f_{k,l,m-1,n,p,q},\\
&(2a_5+2n-1)(-a_4-a_5-2b_4+n-p+1)\, f_{k,l,m,n,p,q} = (-2a_5+2n-1)(a_5+a_6+2b_3-m+n-2)\, f_{k,l,m,n-1,p,q},\\
&(2a_4+2p-1)(-2a_e-a_3-a_4+p-q+1)\, f_{k,l,m,n,p,q} = (-2a_4+2p-1)(a_4+a_5+2b_4-n+p-2)\, f_{k,l,m,n,p-1,q},\\
&(2a_3+2q-1)(-4a_2-2a_3+2q+1)\, f_{k,l,m,n,p,q} = 2(-2a_3+2q-1)(2a_e+a_3+a_4-p+q-2)\, f_{k,l,m,n,p,q-1}.
\end{aligned}$$

Straightforwardly solved in terms of hypergeometric functions:

$$F_1(a_j,b_k) = \frac{\left(\frac{3}{2}-a_8\right)_{k-1}\left(2a_1+a_8+\frac{1}{2}\right)_{k-1}\left(\frac{3}{2}-a_7\right)_{l-1}\left(\frac{3}{2}-a_6\right)_{m-1}\left(\frac{3}{2}-a_5\right)_{n-1}\left(\frac{3}{2}-a_4\right)_{p-1}\left(\frac{3}{2}-a_3\right)_{q-1}(a_7+a_8+2b_1-1)_{l-1}(a_6+a_7+2b_2-1)_{m-1}(a_5+a_6+2b_3-1)_{n-1}(a_4+a_5+2b_4-1)_{p-1}(-p+a_3+a_4+2a_e)_{q-1}}{\left(a_8+\frac{3}{2}\right)_{k-1}\left(a_7+\frac{3}{2}\right)_{l-1}\left(a_6+\frac{3}{2}\right)_{m-1}\left(a_5+\frac{3}{2}\right)_{n-1}\left(a_4+\frac{3}{2}\right)_{p-1}\left(-2a_2-a_3+\frac{5}{2}\right)_{q-1}\left(a_3+\frac{3}{2}\right)_{q-1}(-a_3-a_4-2a_e+2)_{p-1}(-l-a_7-a_8-2b_1+3)_{k-1}(-m-a_6-a_7-2b_2+3)_{l-1}(-n-a_5-a_6-2b_3+3)_{m-1}(-p-a_4-a_5-2b_4+3)_{n-1}}$$

Can fix linear combination of basis functions algorithmically: [Ferrando, FL Mierau, Stawinski in progress]

$$\phi = \sum_k A_k F_k(a_j, b_k)$$

$\widehat{\mathrm{P}}$ symmetries determine all cases we investigated (trees in 1D) .

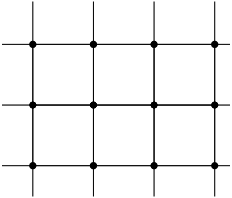
**Summary:**

- Planar Feynman graphs are $\widehat{\mathrm{P}}^{\mu}$ invariant
  (all position-space trees, constraints at loop order)
- Fishnets not 'special' (in this respect)
- Proves and generalizes previous findings beyond integrability
- $\widehat{\mathrm{P}}$ symmetries define 1D track integrals

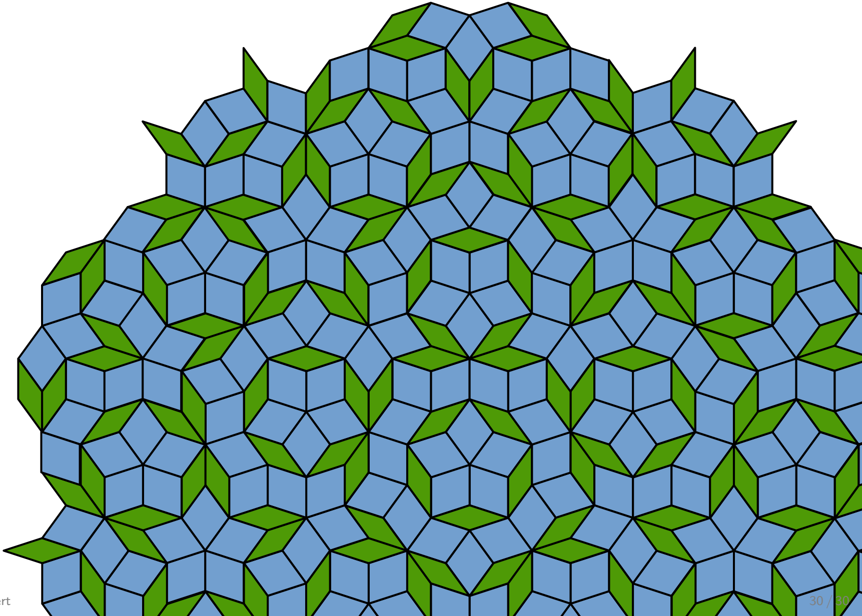
**Outlook:**

- Massive $\widehat{\mathbb{P}}$ symmetry looks like a fishnet limit of Coulomb branch
  $\mathcal{N} = 4$ SYM theory [FL, Miczajka 2020] $\rightarrow$ cf. [Kostya's talk]
- Supersymmetrize conformal simplex/mesh approach
- Define Feynman integrals via $\widehat{\mathrm{P}} = f^{\mathrm{P}}{}_{bc}\mathrm{J}^c\mathrm{J}^b$ symmetries
  (cf. conformal Casimirs $C = \kappa_{ab}\mathrm{J}^a\mathrm{J}^b$ and partial waves)

# More Fishnets?



# Natural $\widehat{\mathrm{P}}$ Fishnets



## Natural $\widehat{\mathrm{P}}$ Fishnets