Fishnets and differential equations

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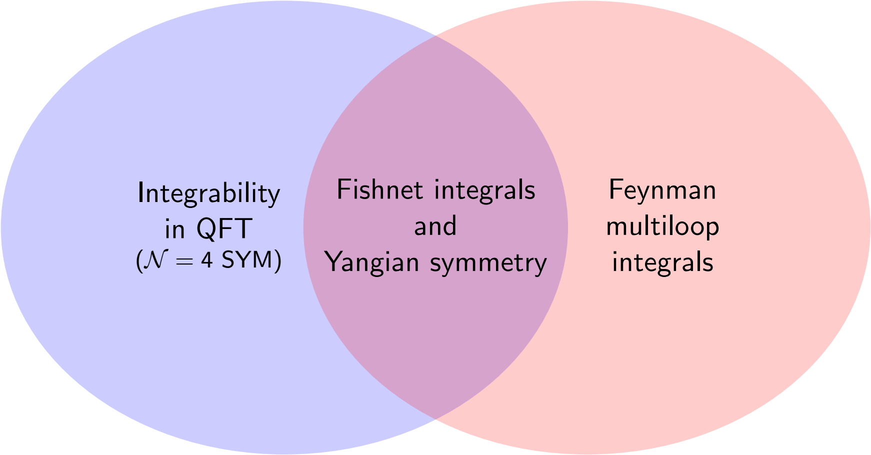
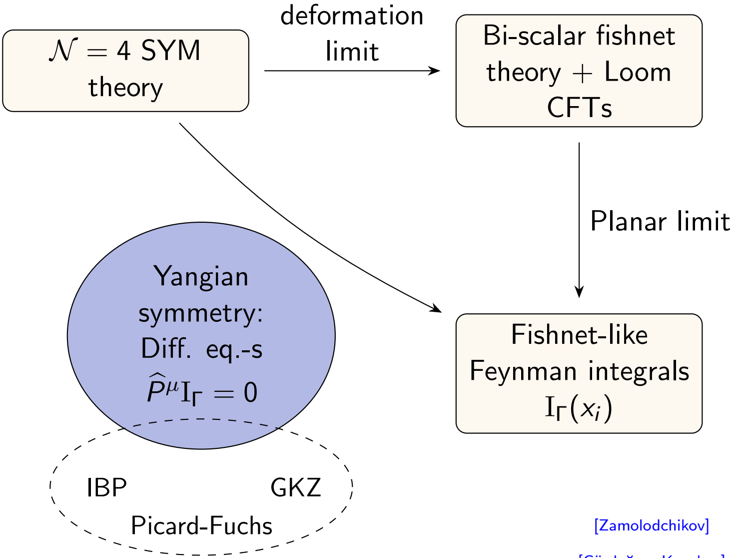
Nordita

Fishnets, Southampton, 2025


Integrability
in QFT
($\mathcal{N} = 4$ SYM)
Fishnet integrals
and
Yangian symmetry
Feynman
multiloop
integrals



$\mathcal{N} = 4$ SYM theory

deformation limit

Bi-scalar fishnet theory + Loom CFTs

Planar limit

Fishnet-like Feynman integrals $\mathrm{I}_{\Gamma}(x_i)$

Yangian symmetry: Diff. eq.-s $\widehat{P}^{\mu}\mathrm{I}_{\Gamma} = 0$

IBP

GKZ

Picard-Fuchs

[Many people]

[Zamolodchikov]

[Gürdoğan, Kazakov]

[Chicherin, Kazakov, Loebbert, Müller, Zhong]

- One corner of multi-loop calculations are differential equations satisfied by Feynman integrals. Effective computationally, good for classification and revealing hidden structures

- Productive approach - study special families: bananas/sunsets, traintracks, <u>fishnets</u>, . . .

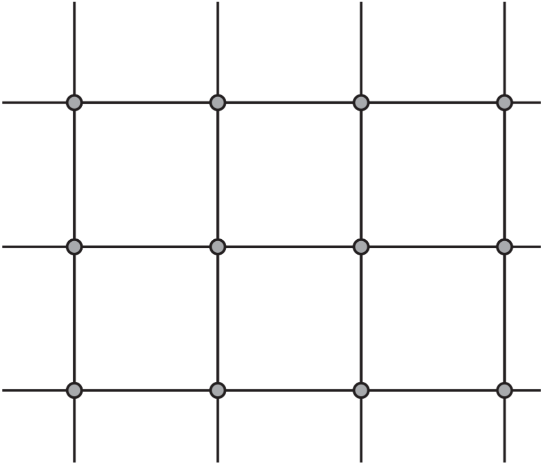
≈ Outline

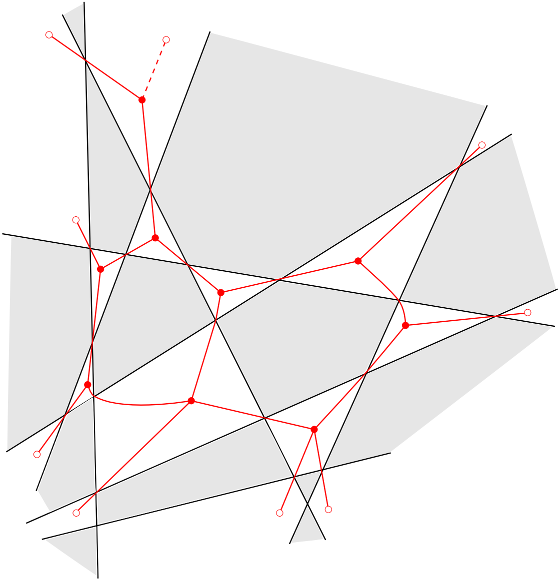
- Fishnet integrals and Yangian invariance.
- Differential equations for multi-loop integrals
- Where is the place for Yangian PDE's?
- Solving and understanding Yangian PDE's

Fishnet Feynman integrals

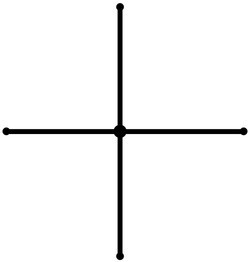
$$\mathrm{I}_\Gamma(x_{ext}|D,\Delta) = \int \prod_{k\in\text{internal}} d^D x_k \prod_{\langle i,j\rangle} \frac{1}{x_{ij}^{2\Delta_{ij}}}$$

- For each vertex $\sum_j \Delta_{ij} = D$
- Planar Γ, special topology (Loom construction [VM, Levkovich-Maslyuk, Kazakov]), extra restrictions on Δ's

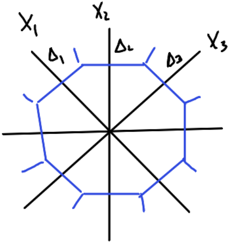




Graph drawn on a Baxter lattice



- The x_i^μ variables can be treated as position space, but also as dual momentum coordinates $x_i^\mu = p_i^\mu - p_i^\mu$
- In this case, we have the momentum space integral of the dual graph.
- The one-loop polygon:



Yangian invariance

- Conformal $\mathfrak{so}(D+1,1)$ symmetry can be represented as:

$$P_j^\mu = -i\partial_{x_j^\mu}, \quad D_j = x_j^\mu \partial_{x_j^\mu} - i\Delta_j, \quad L_j^{\mu\nu} = \dots, \quad K_j^\mu = \dots$$

$$P^\mu \cdot \mathrm{I}_\Gamma\left(D, \Delta_i | x_i\right) = \sum_j P_j^\mu \mathrm{I}_\Gamma\left(D, \Delta_i | x_i\right) = 0$$

- the sum goes over all external vertices.

- Massless integrals are conformal if the sum of propagator dimensions in each vertex is D
- Only required for K^μ. Scale invariance is always there for massless integrals.

Additional symmetry, now we need the extra constraint on Γ:

$$\hat{P}^{\mu}\mathrm{I}_{\Gamma}(D,\Delta|x)=0$$

with

$$\begin{aligned}\widehat{P}^{\mu}=&-\frac{i}{2}\sum_{j<k}[(L_j^{\mu\nu}+g^{\mu\nu}D_j)P_{k,\nu}-(j\leftrightarrow k)]+\sum_j s_j P_j^{\mu}=\\ =&\frac{1}{2}\sum_{j<k}\left(\delta^{\mu\alpha}\delta^{\lambda\nu}-\delta^{\nu\alpha}\delta^{\mu\lambda}-\delta^{\mu\nu}\delta^{\alpha\lambda}\right)(x_j-x_k)^{\alpha}\frac{\partial^2}{\partial x_j^{\lambda}\partial x_k^{\nu}}+\\ &+\sum_j s_j\frac{\partial}{\partial x_j^{\mu}}\end{aligned}$$

where P_j^{μ} etc. act on the j'th external leg and parameters $s_j(\Gamma)$ depend on the graph.

- Yangian algebra $Y(\mathfrak{g})$ [Drinfield] , quantization of $\mathfrak{g}[u]$
- Generators: J^a - level 0, $\hat{J}^a$ - level 1.
- Relations:

$$[\mathrm{J}^a, \mathrm{J}^b] = f_c^{ab}\mathrm{J}^c \qquad [\hat{\mathrm{J}}^a, \mathrm{J}^b] = f_c^{ab}\hat{\mathrm{J}}^c$$

$$\left[\hat{\mathrm{J}}_a, \left[\hat{\mathrm{J}}_b,\ \mathrm{J}_c\right]\right] - \left[\mathrm{J}_a, \left[\hat{\mathrm{J}}_b, \hat{\mathrm{J}}_c\right]\right] \sim \left(\mathrm{J}^3\right). \quad \vdots$$

- Yangian algebra $Y(\mathfrak{g})$ [Drinfield] , quantization of $\mathfrak{g}[u]$
- Generators: J^a - level 0, $\hat{J}^a$ - level 1.

$$\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}_k^a,$$

$$\hat{\mathrm{J}}^a = f^a_{bc} \sum_{j<k=1}^{n} \mathrm{J}_j^c \; \mathrm{J}_k^b + \sum_{k=1}^{n} s_k \; \mathrm{J}_k^a,$$

- Yangian algebra $Y(\mathfrak{g})$ [Drinfield] , quantization of $\mathfrak{g}[u]$
- Generators: J^a - level 0, $\hat{J}^a$ - level 1.
- In our case $\mathfrak{g} = \mathfrak{so}(D+1,1)$. In
- Hence we are solving Yangian invariance for an infinite dimensional representation.

Equations for Feynman integrals

- For the moment with masses and in momentum space, in general D.

Integration by parts identities [K. Chetyrkin, F. Tkachev (1981)]

$$0 = \int \prod_{l \in \text{loop}} d^D q_l \, \frac{\partial}{\partial q_n^\mu} q_{\text{IPB}}^\mu \prod_{\langle i,j \rangle} \frac{1}{\Big((k_{ij}(p_{ext}, q))^2 - m_{ij}^2 \Big)^{\nu_{ij}}} = \sum_{\vec{\nu}'} \mathrm{I}_\Gamma(\ldots, \vec{\nu}')$$

- Using IPB one can express all integrals in terms of a few *master integrals* I_i
- Differentiation of a master integral over external momenta produces again a sum over master integrals with shifted ν parameters and sometimes contracted lines

$$p^{\mu}_{ext} \frac{\partial}{\partial p^{\mu}_{ext}} \mathrm{I}_i = \sum_j A_{ij} \mathrm{I}_j$$

Schwinger/Feynman parameters and Gelfand-Kapranov-Zelevinsky equations:

- Introduce Schwinger parameters

$$\frac{1}{(p^2 - m^2)^\nu} = \int_0^\infty d\alpha\, \alpha^{\nu-1} e^{\alpha(p^2-m^2)}$$

Momentum space integral is now Gaussian and can be taken

- The integral over parameters is then:

$$\int_0^\infty \prod_{i=1}^{n_E} d\alpha_i\, \alpha_i^{\nu_i-1} \delta\left(1-\sum_{i=1}^{n}\alpha_i\right) \frac{F^{LD/2-\sum\nu_i}}{U^{D/2(L+1)-\sum\nu_i}}$$

- Or after further transformation in the Lee-Pomeransky representation [Lee, Pomeransky] :

$$\int_0^\infty \prod_{i=1}^{n_E} d\alpha_i\, \alpha_i^{\nu-1}\,(F+U)^{\nu_0}$$

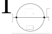
- F and U are the Symanzik polynomials in α_i's, with coefficients made from masses and external momenta (invariants).

- For example, one-loop bubble:

$$
\begin{aligned}
U &= \alpha_1 + \alpha_2 \\
F &= m_1^2\alpha_1^2 + m_2^2\alpha_2^2 + (p^2 + m_1^2 + m_2^2)\alpha_1\alpha_2
\end{aligned}
$$

Picard-Fuchs equations

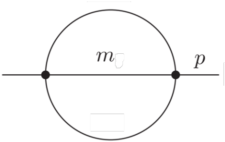
- Maximal cut of the sunset graph in $D = 2$:



$$\mathrm{I}(t = p^2) := \oint \frac{d\alpha_1 d\alpha_2 d\alpha_3}{F}$$

Where

$$F = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1\alpha_2 + \alpha_3\alpha_2 + \alpha_1\alpha_3)\, m^2 - t\alpha_1\alpha_2\alpha_3$$



- Elliptic curve ⇒ two-dim cohomology ⇒ PF equation [Lairez, Vanhove] :

$$\left(t(t-m^2)(t-9m^2)\frac{d^2}{dt^2}+\right.$$

$$\left.+(3t^2-20m^2t+9m^4)\frac{d}{dt}+(t-3m^2)\right)\mathrm{I}$$

$$(t)=0$$

- One can find (one of the solutions):

$$\mathrm{I}$$

$$(t)=\sum_{n=1}^{\infty}\left(\frac{3^{1-2n}+5}{16n}\right)t^n=\frac{1}{16}\left(-5\log\left(1-\frac{t}{m^2}\right)-3\log\left(1-\frac{t}{9m^2}\right)\right)$$

- Equations represent the geometry. Multiloop sunset graphs - CY $\ell-1$-folds [Bönisch, Duhr, Fischbach, Klemm, Nega],[de la Cruz, Vanhove],[Lairez, Vanhove]

GKZ equations

- Take the one-loop bubble in Lee-Pomerasnky rep:

$$\mathrm{I}_{\mathsf{bubble}}(m_1^2, m_2^2, p^2) = \int (U + F)^{-D/2} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2$$

with the Symanzik polynomials

$$\begin{aligned} U &= \alpha_1 + \alpha_2 \\ F &= m_1^2 \alpha_1^2 + m_2^2 \alpha_2^2 + (p^2 + m_1^2 + m_2^2) \alpha_1 \alpha_2 \end{aligned}$$

$$U + F = \alpha_1 + \alpha_2 + m_1^2 \alpha_1^2 + m_2^2 \alpha_2^2 + (p^2 + m_1^2 + m_2^2) \alpha_1 \alpha_2$$

- Lift the polynomials to generic coefficients.

$$\mathrm{I}_{\mathsf{GKZ}}(z) = \int d\alpha_1 d\alpha_2 \alpha_1^{\nu_1}\alpha_2^{\nu_2}\left(z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_1^2 + z_4\alpha_1\alpha_2 + z_5\alpha_2^2\right)^{-D/2}$$

It satisfies [Gel'fand, Zelevinskii, Kapranov]

$$\left(\frac{\partial^2}{\partial z_4^2} - \frac{\partial^2}{\partial z_3 \partial z_5}\right)\mathrm{I}_{\mathsf{GKZ}}(z) = 0\,,\ \left(\frac{\partial^2}{\partial z_1 \partial z_4} - \frac{\partial^2}{\partial z_3 \partial z_2}\right)\mathrm{I}_{\mathsf{GKZ}}(z) = 0$$

- $+$ scaling: $z_1\partial_1 + 2z_3\partial_3 + z_4\partial_4 + (1+\nu_1)\mathrm{I}_{\mathsf{GKZ}}(z) = 0$, . . .
- Such equations are obeyed by roots of polynomial equations, periods of toric Calabi-Yau manifolds, and generic Euler type integrals

- Lift the polynomials to generic coefficients.

$$\mathrm{I}_{\mathsf{GKZ}}(z) = \int d\alpha_1 d\alpha_2 \alpha_1^{\nu_1}\alpha_2^{\nu_2}\left(z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_1^2 + z_4\alpha_1\alpha_2 + z_5\alpha_2^2\right)^{-D/2}$$

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- Solutions - generalized hypergeomemetric series, $\mathcal{A}$-hypergeometric functions
- F.I. also lie in this class of functions [E Nasrollahpoursamami (2016), K. Schultka (2018), P. Vanhove (2018), L. de la Cruz (2019)] :

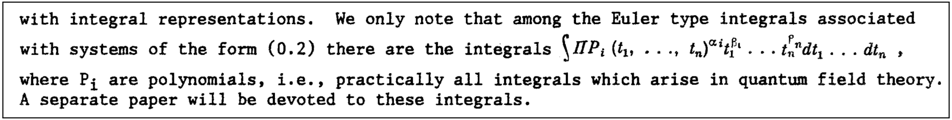
$$\begin{aligned}&\mathrm{I}_{\mathsf{bubble}}(m_1^2, m_2^2, p^2) = \\ =&\mathrm{I}_{GKZ}(a_{1,0} = 1, a_{0,1} = 1, a_{2,0} = m_1^2, a_{2,0} = m_2^2, a_{1,1} = m_1^2 + m_2^2 + p^2)\end{aligned}$$

- Lift the polynomials to generic coefficients.

$$\mathrm{I}_{\mathsf{GKZ}}(z) = \int d\alpha_1 d\alpha_2 \alpha_1^{\nu_1} \alpha_2^{\nu_2} \left(z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_1^2 + z_4\alpha_1\alpha_2 + z_5\alpha_2^2 \right)^{-D/2}$$

It satisfies [Gel'fand, Zelevinskii, Kapranov]

$$\left(\frac{\partial^2}{\partial z_4^2} - \frac{\partial^2}{\partial z_3 \partial z_5} \right) \mathrm{I}_{\mathsf{GKZ}}(z) = 0\,, \; \left(\frac{\partial^2}{\partial z_1 \partial z_4} - \frac{\partial^2}{\partial z_3 \partial z_2} \right) \mathrm{I}_{\mathsf{GKZ}}(z) = 0$$



with integral representations. We only note that among the Euler type integrals associated with systems of the form (0.2) there are the integrals $\int \Pi P_i\,(t_1, \ldots, t_n)^{\alpha_i} t_1^{\beta_1} \ldots t_n^{\rho_n} dt_1 \ldots dt_n$, where P_i are polynomials, i.e., practically all integrals which arise in quantum field theory. A separate paper will be devoted to these integrals.

Equations for fishnet integrals

- Fishnet integrals are Yangian invariant, that is we have equations:

$$\left(\frac{1}{2}\sum_{j<k}\left(\delta^{\mu\alpha}\delta^{\lambda\nu}-\delta^{\nu\alpha}\delta^{\mu\lambda}-\delta^{\mu\nu}\delta^{\alpha\lambda}\right)(x_j-x_k)^{\alpha}\frac{\partial^2}{\partial x_j^{\lambda}\partial x_k^{\nu}}+\right.$$
$$\left.+\sum_j s_j\frac{\partial}{\partial x_j^{\mu}}\right)\mathrm{I}_{\Gamma}(x)=0$$

- $+$ conformal symmetry:

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i^{\mu}}\,\mathrm{I}_{\Gamma}(x)=0$$
$$\left(\sum_{i=1}^{n}x_i^{\mu}\frac{\partial}{\partial x_i^{\mu}}+\Delta_i\right)\mathrm{I}_{\Gamma}(x)=0$$
$$\vdots$$

- Conformal symmetry implies that:

$$\mathrm{I}_\Gamma(x) = \prod_{i<j} x_{ij}^{2\beta_{ij}} \mathrm{I}_\Gamma^{(0)}(\xi^A)$$

Cross ratios:

$$\xi^A = \prod_{i<j} x_{ij}^{2\alpha_{ij}}$$

with

$$\alpha_{ij}^A = \alpha_{ji}^A\,,\ \alpha_{ii}^A = 0\,, \sum_i \alpha_{ij}^A = 0$$

Where A labels different $\frac{N(N-3)}{2}$ cross ratios $\left(ND - \frac{(D+1)(D+2)}{2}\right)$

- Conformal weights satisfy

$$\beta_{ij} = \beta_{ji} \ \ \beta_{ii} = 0 \sum_i \beta_{ij} = -\Delta_i$$

- The level one generator then rewrites in terms of cross ratios as [F.Loebbert, D.Müller, H.Münkler] :

$$\hat{P}^{\mu} = \sum_{jk} \frac{x_{jk}^{\mu}}{x_{jk}^{2}} \mathrm{PDE}_{jk}$$

- Equations PDE_{jk} are purely in terms of cross ratios

- Example - the cross [F.Loebbert, D.Müller, H.Münkle]

$$\mathrm{I}_+(x) = \int \frac{d^D x_0}{x_{10}^{2\Delta_1} x_{20}^{2\Delta_2} x_{30}^{2\Delta_3} x_{40}^{2\Delta_4}} =$$

$$= x_{14}^{2\Delta_2+2\Delta_3-D} x_{13}^{2\Delta_4-D} x_{34}^{-2\Delta_3-2\Delta_4+D} x_{24}^{-2\Delta_2} \mathrm{I}_+^{(0)}(u,v)$$

- Cross ratios are chosen as:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$\alpha_{12}^1 = \alpha_{34}^1 = -\alpha_{13}^1 = -\alpha_{24}^1 = 1$$
$$\alpha_{14}^2 = \alpha_{23}^1 = -\alpha_{13}^1 = -\alpha_{24}^1 = 1$$

- Out of $4 \cdot 3/2 = 6\ \mathrm{PDE}_{ik}$ only 2 are independent

$$\begin{aligned} 0 = \big(&\alpha\beta + (\alpha+\beta)(u\partial_u + v\partial_v) + \\ &+(u\partial_u + v\partial_v)^2 - u\partial_u^2 - \gamma\partial_u\big)\, I_+^{(0)}(u,v) \\ 0 = \big(&\alpha\beta + (\alpha+\beta)(u\partial_u + v\partial_v) + \\ &+(u\partial_u + v\partial_v)^2 - v\partial_v^2 - \gamma'\partial_v\big)\, I_+^{(0)}(u,v). \end{aligned}$$

- 4-dim solution space - Appel F_4 functions

- $+$ choice of convergence region $+$ symmetries $+$ boundary conditions [F.Loebbert, D.Müller, H.Münkler]

- In $D = 2$ and $\Delta_i = 1/2$, change to coordinates

$$z\bar{z} = u, \quad (1 - z)(1 - \bar{z}) = v.$$

Then the integral is:

$$\mathrm{I}_+(x) = \frac{1}{|x_{12}|\,|x_{34}|})I_+^{(0)}(z, \bar{z}).$$

- The equations factorize into a holomorphic and anti-hol. part:

$$\left(1 + 4(2z - 1)\partial_z + 4z(z - 1)\partial_z^2\right) I_+^{(0)}(z, \bar{z}) = 0$$

- $I_+^{(0)}(z, \bar{z})$ - expressed via periods of an elliptic curve.

- For rectangular fishnets [Duhr, Klemm, Loebbert, Nega, Porkert] :

$$\mathrm{I}_\Gamma(z) = \int \left(\prod_{j=1}^{\ell} \frac{\mathrm{d}x_j \wedge \, \mathrm{d}\bar{x}_j}{-2i} \right) \frac{1}{\sqrt{|P_\Gamma(x,z)|^2}}$$

- Expressed in terms of periods of the variety:

$$y^2 = P_\Gamma(x,z)$$

  With the degrees of P_Γ being exactly such that they define CY manifold. ℓ-loop $=$ ℓ-fold.

- The Yangian equations PDE_{ik} - Picard-Fuchs equations for these CY.

- Generalizations . . .

- General form of the equation [VM, Levkovich-Maslyuk]

$$\mathrm{PDE}_{ik} = 2\left(\sum_{l>j>i} - \sum_{l<j<i} + \sum_{l<k<i,j} - \sum_{l>k>i,j}\right)\chi_{iklj}\theta_{il}\theta_{jk} +$$

$$+\sum_{j\neq i}(\delta_{j>i}-\delta_{j<i})\theta_{ik}\theta_{ij} + \delta_{i>k}\left(2\sum_{j=k+1}^{i-1}\Delta_j + \Delta_i + D\right)\theta_{ik} -$$

$$-\,\delta_{i<k}\left(2\sum_{j=i+1}^{k-1}\Delta_j + \Delta_i + D\right)\theta_{ik} + 2(s_k - s_i)\theta_{ik}$$

where:

$$\chi_{iklj} = \frac{x_{ik}^2 x_{lj}^2}{x_{il}^2 x_{kj}^2}\,, \quad \theta_{ij} = \sum_A \alpha_{ij}^A \xi^A \frac{\partial}{\partial \xi^A} + \beta_{ij}$$

- You could solve a different problem: what are the possible Yangian invariant functions?

$$\hat{P}^{\mu}(s_j) \cdot \mathrm{F}(x) = 0$$
$$\mathfrak{so}(D+1,1) \cdot \mathrm{F}(x) = 0$$

- One finds (experimentally [VM, Levkovich-Maslyuk, Kazakov]):

$$s_{j+1} - s_j = -\frac{\Delta_j + \Delta_{j+1}}{2} - \sum_{i=1}^{l} (\tilde{\Delta}_i - D/2)$$

- Agree with the Loom and other approaches [Loebbert, Rüenaufer, Stawinski]

- Disentangle the level-one $\hat{P}^\mu$ symmetry, from conformal symmetry

$$x_i^\mu \longrightarrow x_{ij}^2 \longrightarrow u, v, \ldots$$

- The level-one operator is [VM, Levkovich-Maslyuk, Kazakov] :

$$\hat{P}^\mu \to \sum_{lj} L_{iklj} \quad + \quad \hat{R}_{ik}$$

where

$$L_{iklj} = \frac{\partial^2}{\partial(x_{ik}^2)\partial(x_{lj}^2)} - \frac{\partial^2}{\partial(x_{il}^2)\partial(x_{kj}^2)}$$

[Pal,Ray]

- The level 0, i.e. conformal constraints:

$$\sum_{j \neq i} x_{ij}^2 \frac{\partial}{\partial(x_{ij}^2)} - \Delta_i$$

Towards solving Yangian PDE's

GKZ systems:

$$\mathcal{A} \quad + \quad \vec{b} \quad =$$

$n \times N$ matrix

vector $\mathbb{C}^n$

$$\ell \in \mathrm{ker}\,(\mathcal{A})$$

$$\prod_{\ell_i>0} \frac{\partial^{\ell_i}}{\partial {z_i}^{\ell_i}} - \prod_{\ell_i<0} \frac{\partial^{-\ell_i}}{\partial {z_i}^{-\ell_i}}$$

$$\sum_j \mathcal{A}_{ij} z_j \frac{\partial}{\partial z_j} - b_i$$

system for a function of n-variables z_i

Triangulations,bases,. . . $\downarrow$

$$z_k = x_{ij}^2$$

$\mathcal{A}, \vec{b}$ -

special

Solutions:

$$\sum_{u \in \mathrm{ker}\,\mathcal{A}} \frac{1}{\prod_{i=1}^{N} \Gamma(\gamma_i + u_i + 1)} z_i^{u_i + \gamma_i}$$

$$\mathcal{A}\gamma = b^T$$

- Matrix and vector:

$$\mathcal{A} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \Rightarrow \ker \mathcal{A} = \mathbb{Z} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$$

- Equations:

$$\frac{\partial^2}{\partial z_1^2} - \frac{\partial^2}{\partial z_2 \partial z_3}$$

$$\left.\begin{matrix} z_1\partial_1 + 2z_2\partial_2 - b_1 \\ z_1\partial_1 + 2z_3\partial_3 - b_2 \end{matrix}\right\} \Rightarrow \Phi(z_1, z_2, z_3) = z_2^{\frac{b_1}{2}} z_3^{\frac{b_2}{2}} \Phi_0\left(\frac{z_1^2}{z_2 z_3}\right)$$

$$\Phi_0\left(x = \frac{z_1^2}{z_2 z_3}\right) = c_1\, {}_2F_1\left(-\frac{b_1}{2}, -\frac{b_2}{2}; \frac{1}{2}; \frac{x}{4}\right) + c_2\sqrt{x}\, {}_2F_1\left(\frac{1}{2} - \frac{b_1}{2}, \frac{1}{2} - \frac{b_2}{2}; \frac{3}{2}; \frac{x}{4}\right)$$

- L_{iklj} are a special type of GKZ system - for special type of $\mathcal{A}$ [Gel'fand, Zelevinskii, Kapranov] .
- GKZ equations have known solutions in therms of $\mathcal{A}$-hypergeometric functions and can be treated with hypergeometric methods
- Whenever $\hat{R}$ -vanishes (n-cross, ...) the level-one Yangian is equivalent to GKZ equations
- For our $\mathcal{A}$'s GKZ tell us that the space of solutions is $2^{N-1} - N$ dimensional.

- When does $\hat{R}$ vanish?
- At 4 points we get the conditions:

$$(\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4)(\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4)(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - D) = 0$$

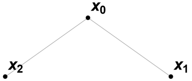
- At 5 points we get many conditions like:

$$\begin{aligned} 2D &= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 \\ D &= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 \\ \Delta_1 + \Delta_4 + \Delta_5 &= \Delta_2 + \Delta_3 \\ D + \Delta_3 &= \Delta_1 + \Delta_2 + \Delta_4 + \Delta_5 \\ &\vdots \end{aligned}$$

More equations (see Florian's talk)

- Two-point "Yangian"':

$$\left(\frac{1}{2}\left(\delta^{\mu\alpha}\delta^{\lambda\nu}-\delta^{\nu\alpha}\delta^{\mu\lambda}-\delta^{\mu\nu}\delta^{\alpha\lambda}\right)(x_1-x_2)^{\alpha}\frac{\partial^2}{\partial x_1^{\lambda}\partial x_2^{\nu}}+ + \Delta_1\frac{\partial}{\partial x_2^{\mu}}-\Delta_2\frac{\partial}{\partial x_1^{\mu}}\right)\cdot$$



$$=0$$

- These are more peculiar equations, that exist depending on the topology, these extend to more intricate relations [[Loebbert,Mathur]]
- Massive fishnets [Loebbert, Miczajka, Müller, Münkler]
- In fact level one symmetry $\hat{P}^{\mu}$ holds independently from the conformal symmetry [Loebbert, Rüenaufer, Stawinski]

Fun things to do with Yangian PDE's:

Restrictions

- Suppose we have an equation for a function $f(x, y)$. Can we find an equation for, say, $g(x) = f(x, 0)$ or $g(x) = f(x, x)$.
- A restriction $\mathcal{D}$-module is well defined for good systems of equations [Henn, Pratt, Sattelberger, Zoia],[Sattelberger, Sturmfels] .
- GKZ systems are *good* - allow restrictions.
- We do that already, when restricting the GKZ systems for Feynman integral to physical parameters.

- Variation 1: Non-conformal integrals

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- Conformal → scale + Poincare. To break constraint $\sum \Delta_i = D$, add an external leg and then send it to infinity.
- By conformal symmetry this is equivalent to $x^2_{(n+1)j} = 1\,,\ \forall j$ Hence $\sum \Delta_i = D - \Delta_{(n+1)}$
- Recall that level-one symmetry is still preserved (the L_{iklj} part of the Yangian GKZ)

- Variation 1: Non-conformal integrals

- Recall the cross:

$$\mathrm{I}_+(x_1, x_2, x_3, x_4) = \int \frac{d^D x_0}{x_{10}^{2\Delta_1} x_{20}^{2\Delta_2} x_{30}^{2\Delta_3} x_{40}^{2\Delta_4}} \qquad x_{ij}^2 = v_{ij}$$

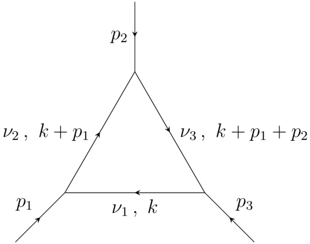
The GKZ operators:

$$\left\langle \frac{\partial^2}{\partial v_{12} \partial v_{34}} - \frac{\partial^2}{\partial v_{13} \partial v_{24}} , \frac{\partial^2}{\partial v_{12} \partial v_{34}} - \frac{\partial^2}{\partial v_{14} \partial v_{23}} \right\rangle$$

$$v_{12}\partial_{12} + v_{13}\partial_{13} + v_{14}\partial_{14} - \Delta_1$$
$$v_{12}\partial_{12} + v_{23}\partial_{23} + v_{24}\partial_{24} - \Delta_2$$
$$\vdots$$

- Variation 1: Non-conformal integrals

- After sending $x_4^\mu \to \infty\,,\ v_{i4} = 0$, we obtain a three point non-conformal integral. In dual momentum coordinates - one-loop triangle



$$y_1 = p_1^2\,,\ y_2 = p_2^2\,,\ y_3 = (p_1+p_2)^2$$

- Variation 1: Non-conformal integrals
- After restriction we have:

$$\left(y_1\partial_1^2 - y_3\partial_3^2\right) + \left(1 - \frac{D}{2} + \Delta_2 + \Delta_3\right)\partial_1 - \left(1 - \frac{D}{2} + \Delta_1 + \Delta_2\right)\partial_3$$

$$\left(y_2\partial_2^2 - y_3\partial_3^2\right) + \left(1 - \frac{D}{2} + \Delta_1 + \Delta_3\right)\partial_2 - \left(1 - \frac{D}{2} + \Delta_1 + \Delta_2\right)\partial_3$$

$$(y_3\partial_3 + y_2\partial_2 + y_1\partial_1) + (\Delta_1 + \Delta_2 + \Delta_3 - D/2)$$

Exactly the equations for the non-conformal triangle/star

- Variation 1: Non-conformal integrals

- From the canonical GKZ approach we get:

$$\mathrm{I}_{\triangle} = \int_{\Omega} \mathrm{d}^3\alpha \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}}{(z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_3 + z_4\alpha_1\alpha_2 + z_5\alpha_1\alpha_3 + z_6\alpha_2\alpha_3)^{\beta}}$$

  The GKZ system is exactly the Yangian one upon certain identifications. Moreover, the restriction to physical variables, requires $z_1 = z_2 = z_3 = 1, z_4 = (p_1 + p_2)^2, z_5 = p_1^2, z_6 = p_2^2$, which is the same restriction.

- Claim: the unrestricted/lifted GKZ system for the non-conformal star/triangle is the Yangian GKZ system for the conformal cross.

- Variation 1: Non-conformal integrals
- Variation 2: Gluing points.

  Restrict to $x_1 = x_2$, or $x_{12}^2 = 0$, $x_{1j}^2 = x_{2j}^2$, for example, from double cross to a simplest ladder.
- Variation 3: The "non-GKZ" Yangian as restriction of a bigger GKZ system.

Some remarks

- Fishnets are interesting diagrams to study from the PDE perspective

- We could potentially write have a complete solution for the Yangian PDE's

- The equations come from an algebra - the Yangian.

- GKZ systems are related to Calabi-Yau geometry. Is our observation related to some D-dimensional deformation of PF equation for fishnet CY periods in $D = 2$?

- Fishnet theory - "completely perturbatively solvable" in some sense?