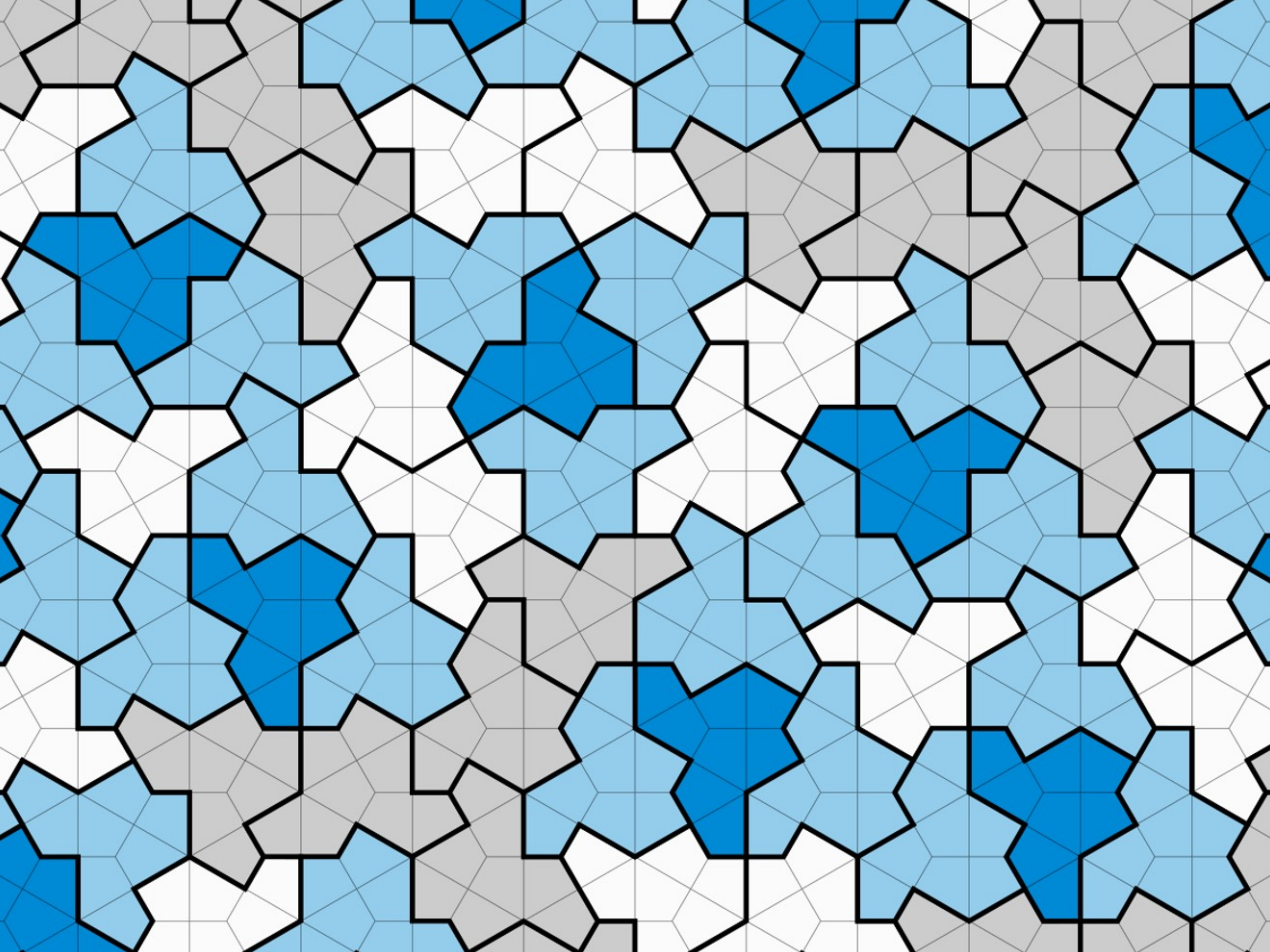
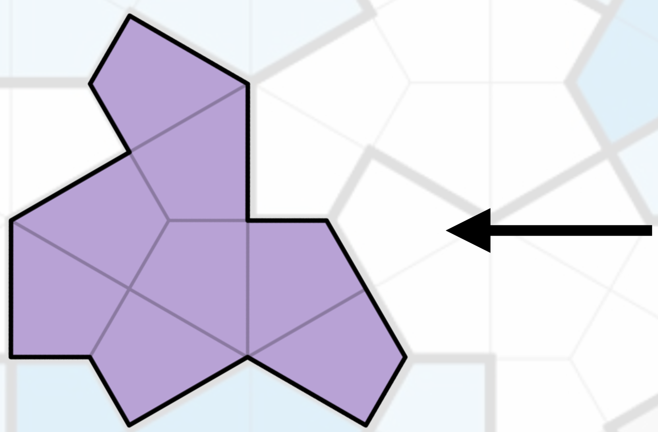
Introduction to Fishnet Spectrum:

A Quantum Spectral Curve approach

Michelangelo Preti

Southampton
14-15th July 2025



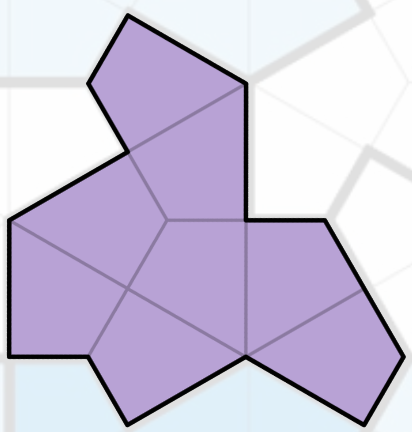


Einstein

[David Smith ’23]

Tessellation

covering of a surface using one or more tiles with no overlaps and no gaps



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Aperiodic Monotile

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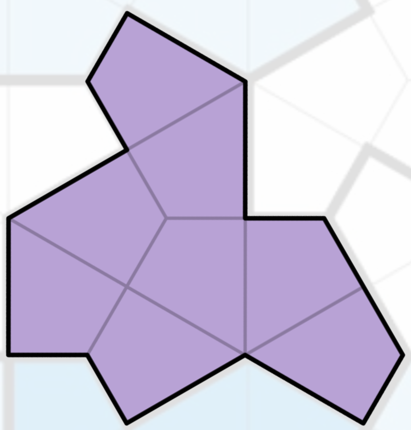
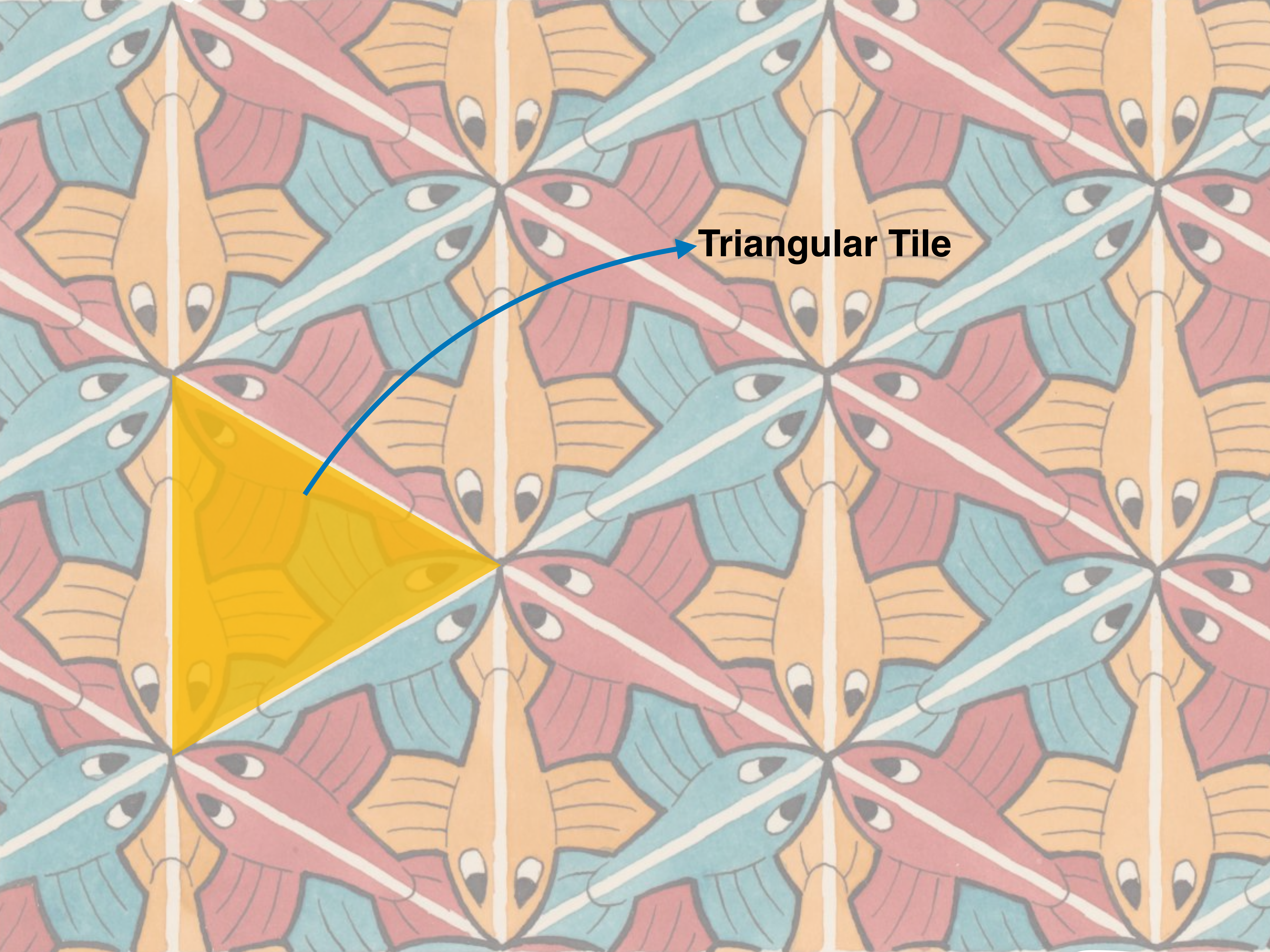


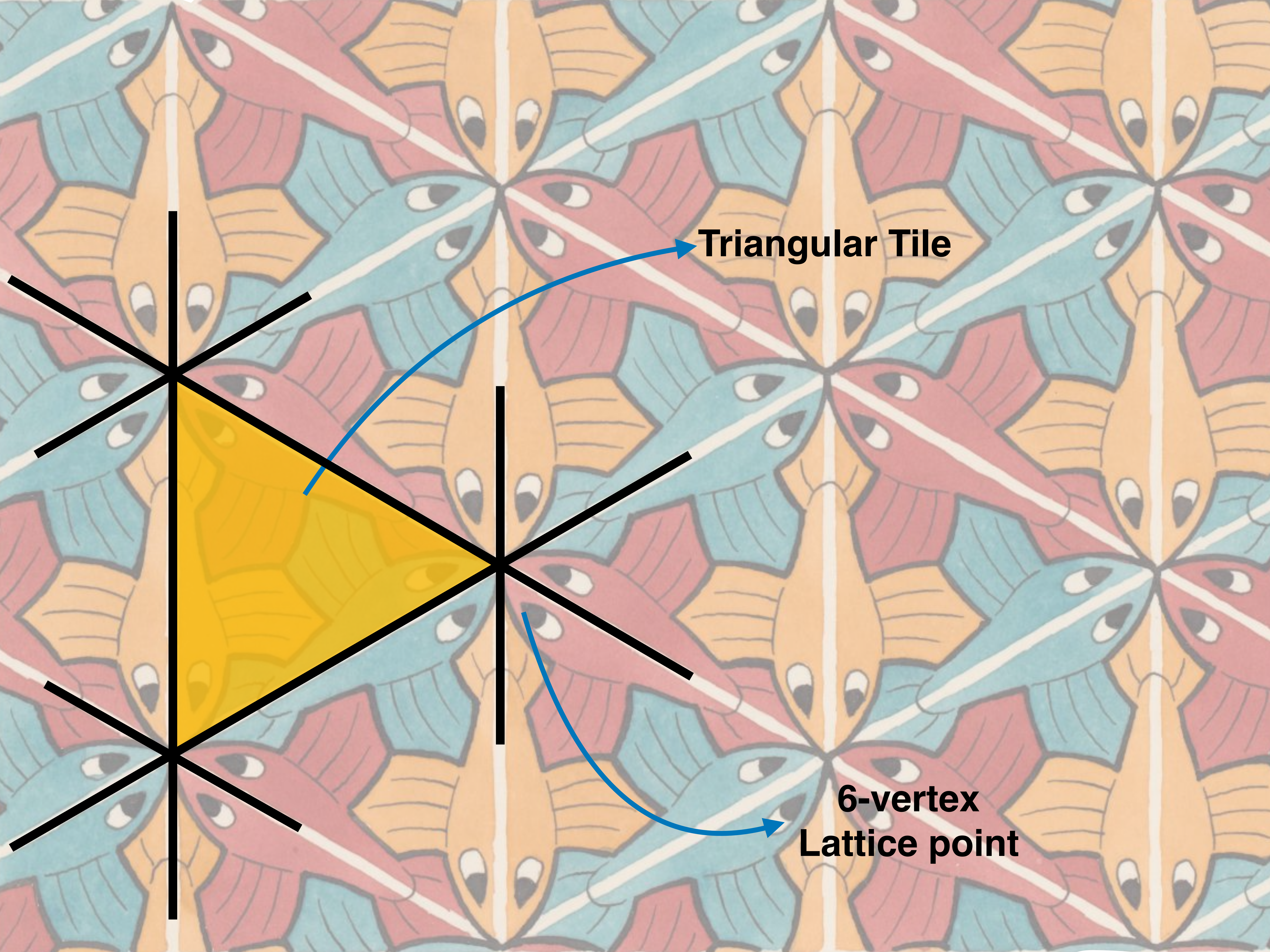
[David Smith ’23]

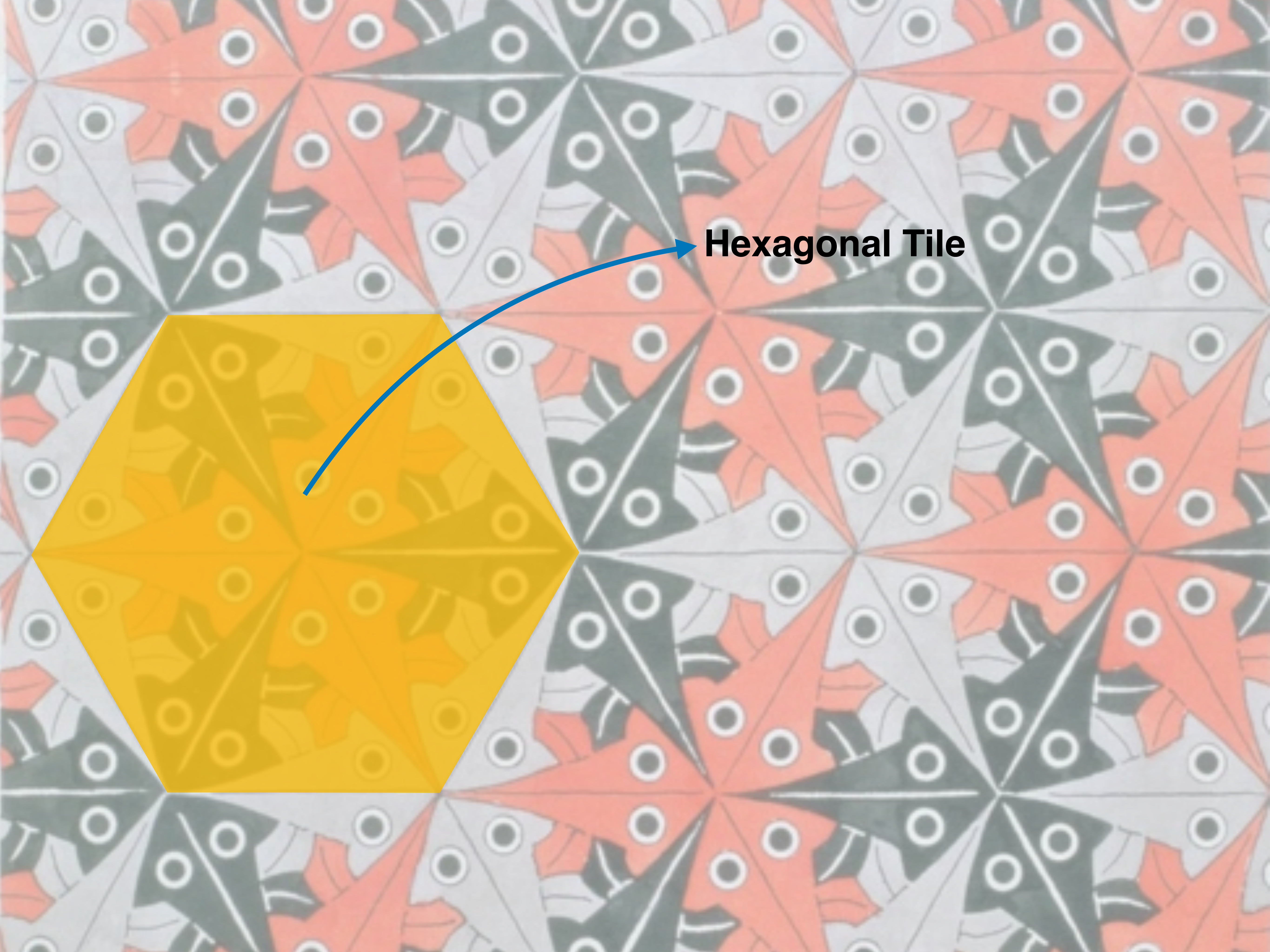
← **Einstein** Aperiodic Monotile

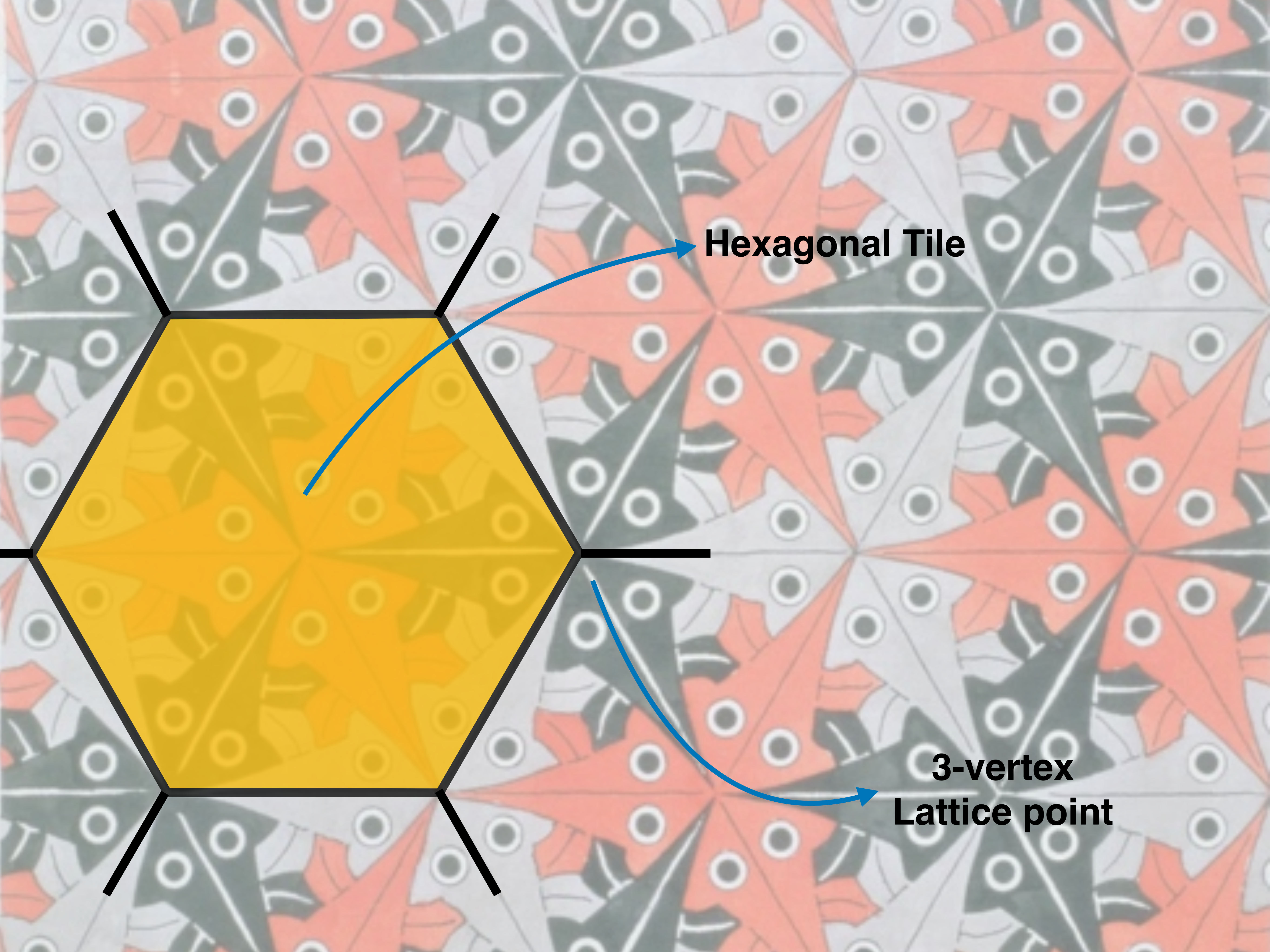
Physicist question:

What are the simplest mono tile tessellation?

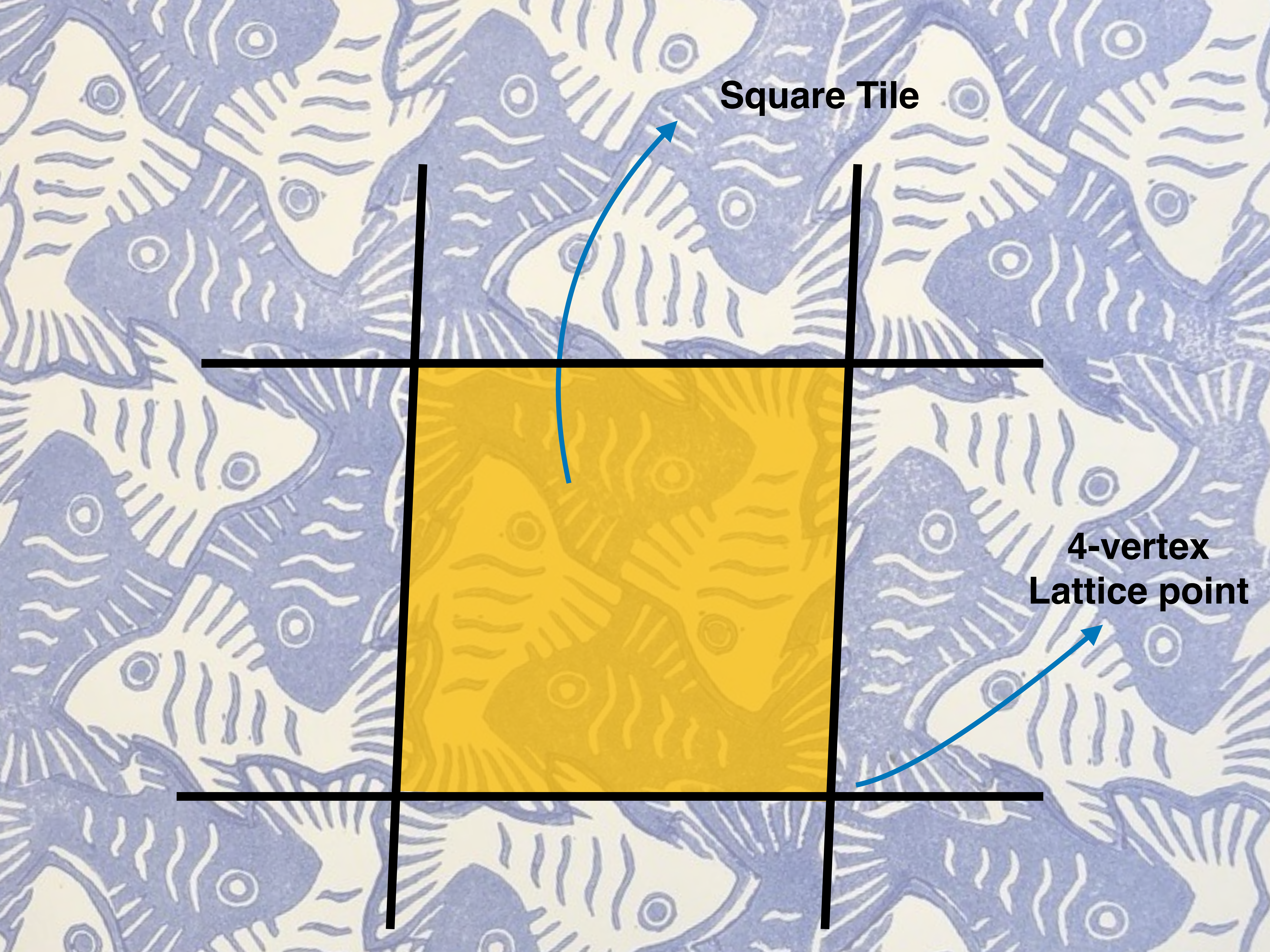

Triangular Tile


Triangular Tile
6-vertex
Lattice point

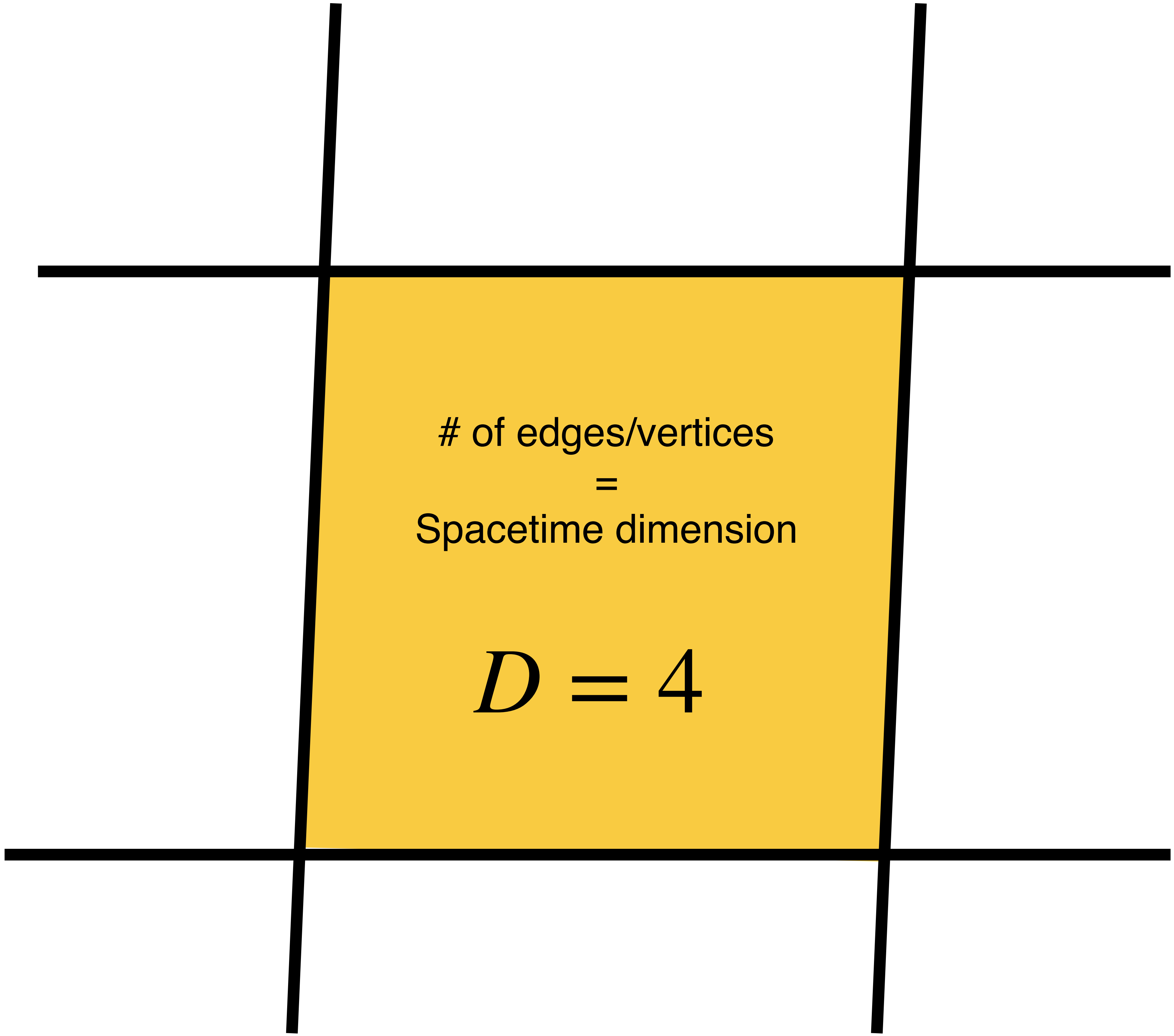

Hexagonal Tile


Hexagonal Tile
3-vertex
Lattice point


Square Tile


Square Tile
4-vertex
Lattice point

From Escher to Fishnet



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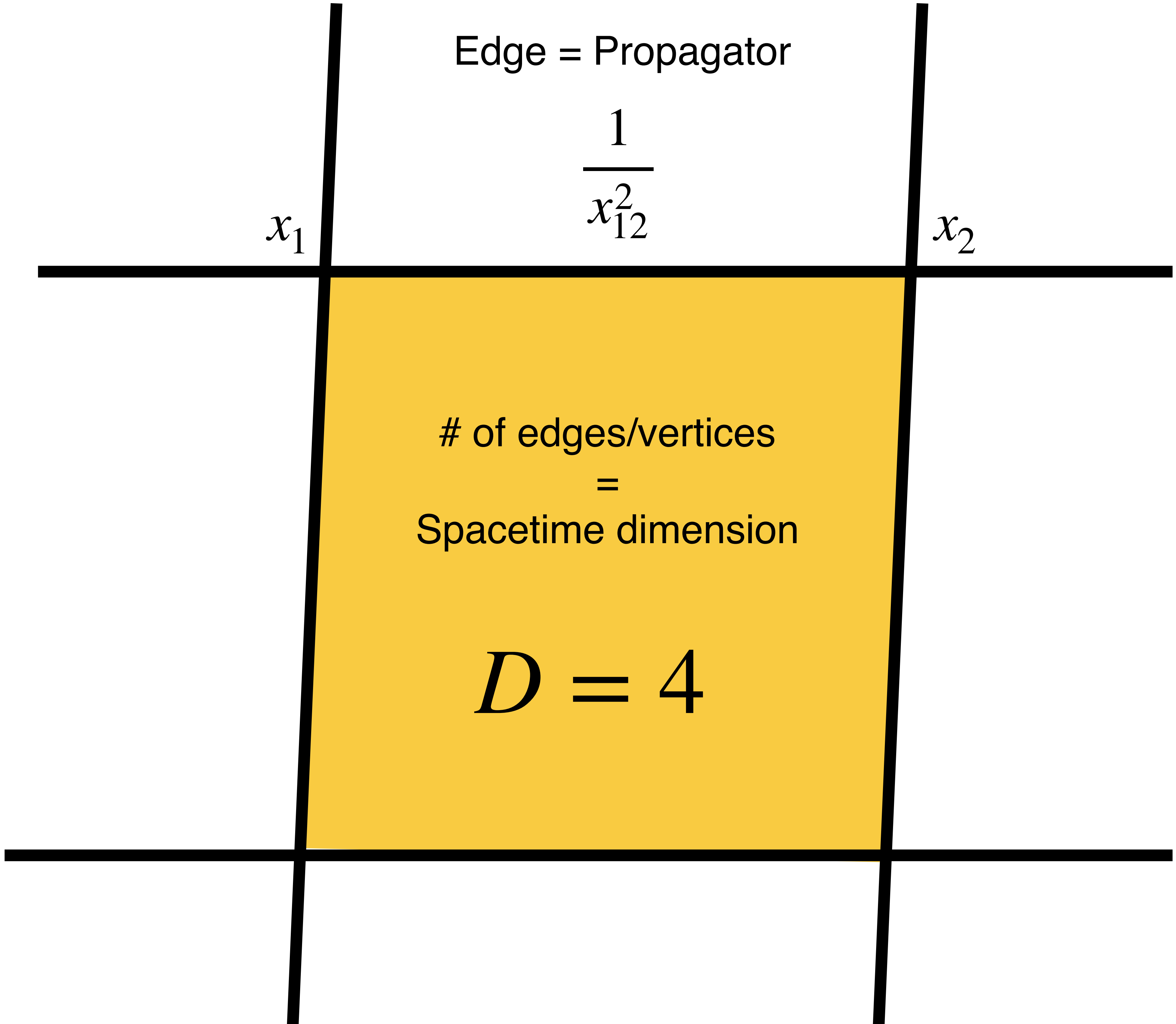

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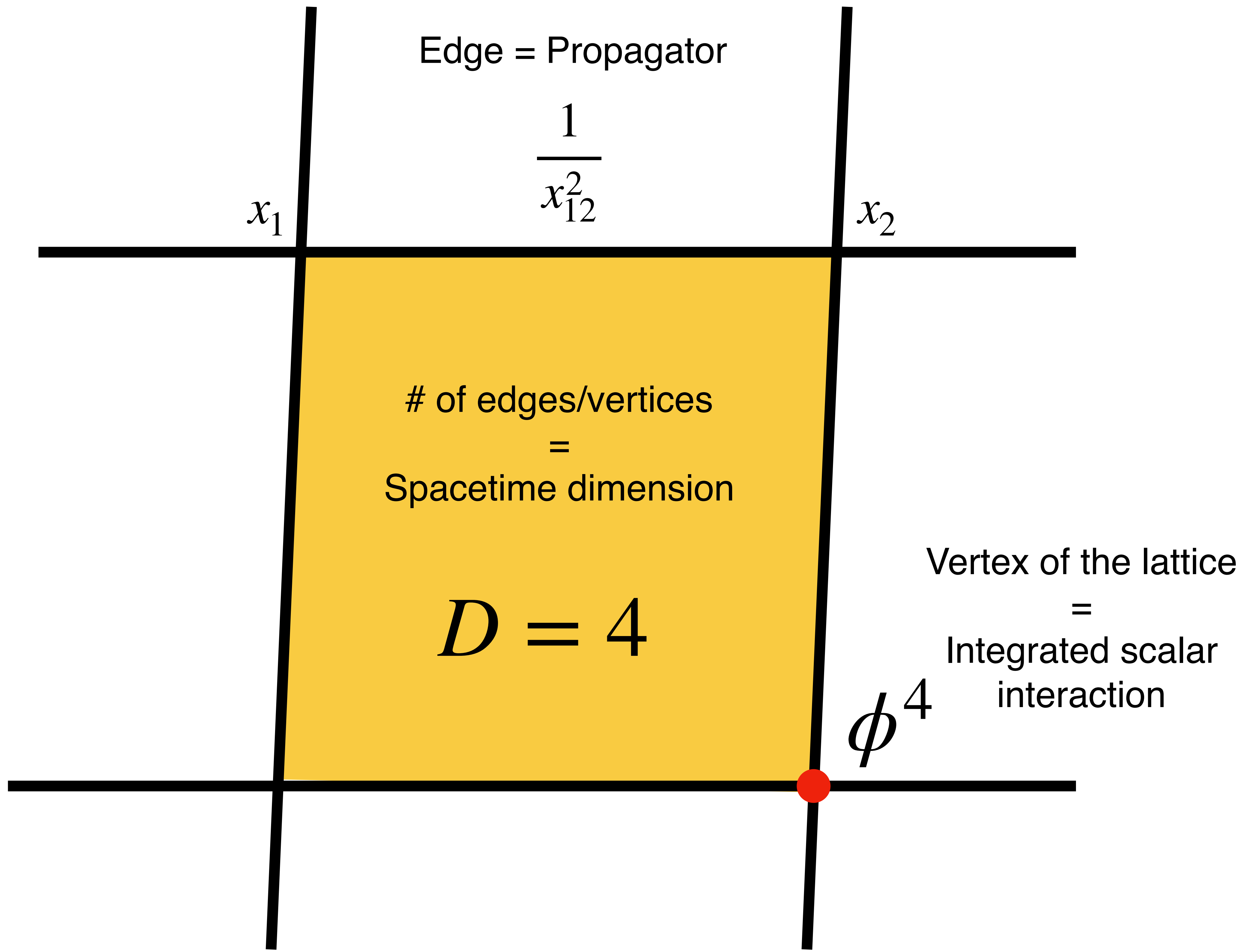
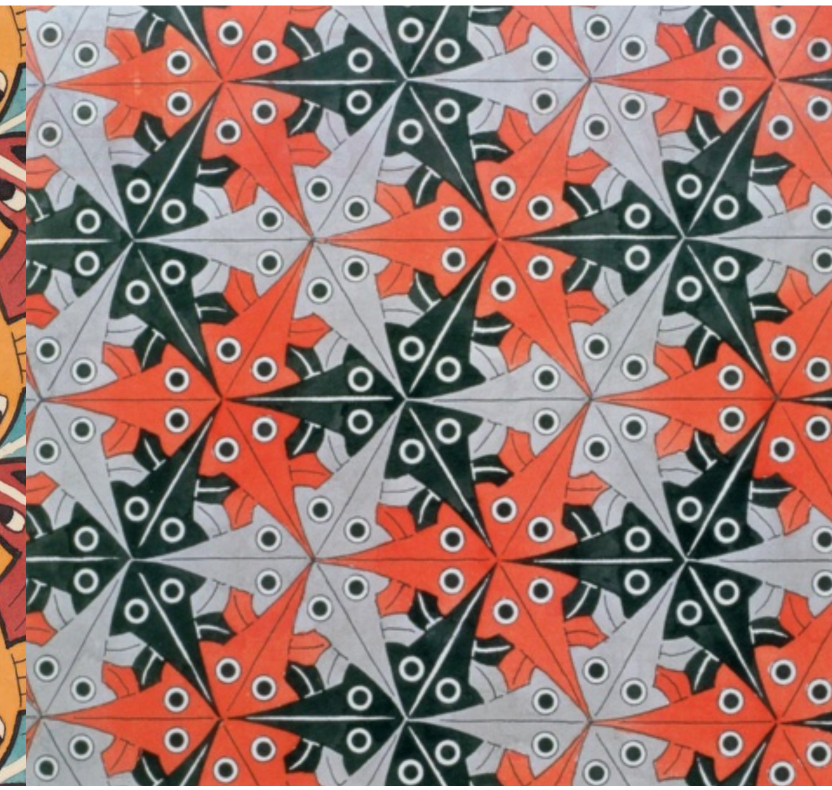


What is a fishnet?

Conformal field theories dominated by a specific class of multi loop Feynman graphs with the topology of a regular lattice

$D = 4 \qquad \phi^4$

$D = 3 \qquad \phi^6$



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[O.Gurdogan,V.Kazakov '16]

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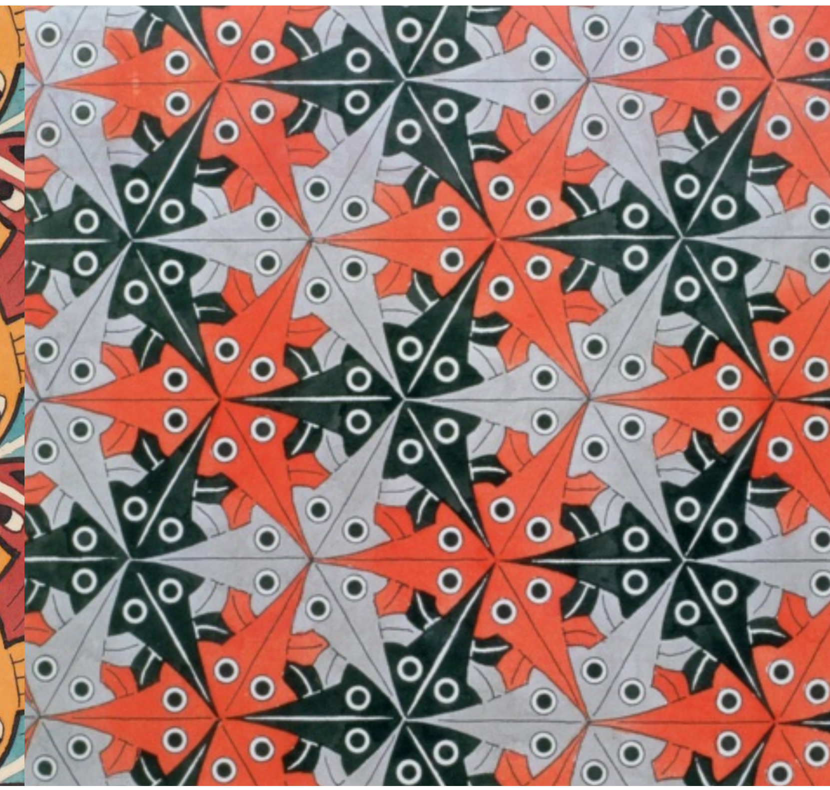

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$(2, 0)$ SCFT ?

[O.Mamroud,G.Torrents '17]

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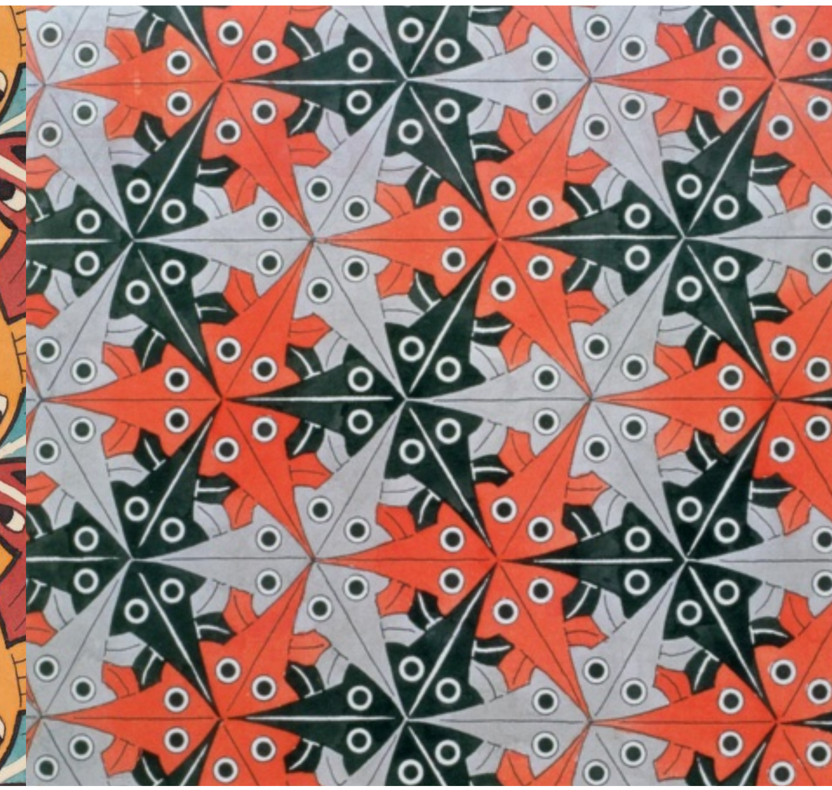

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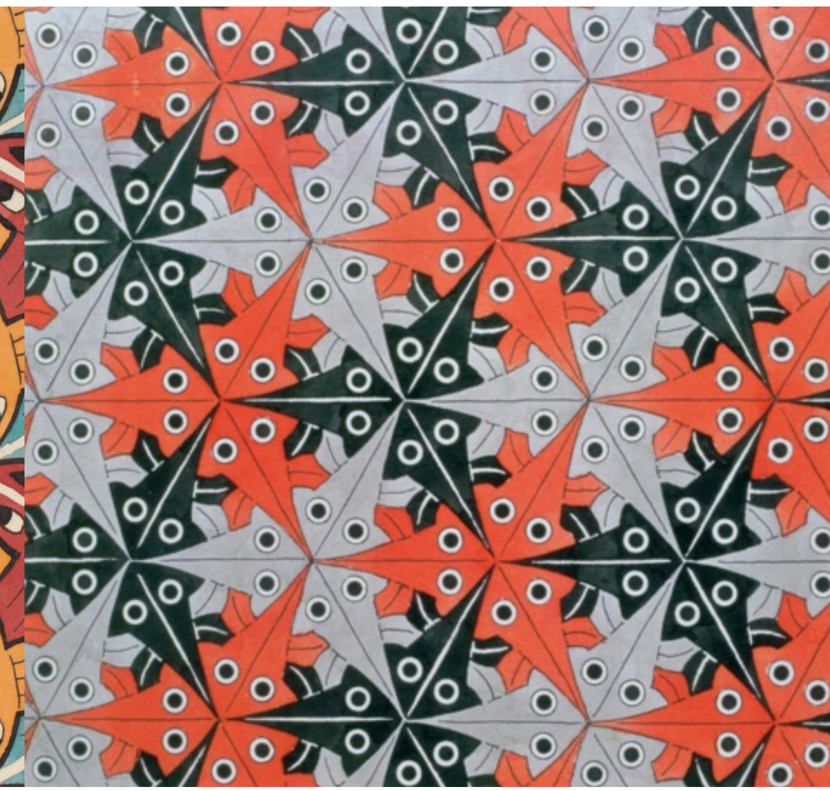

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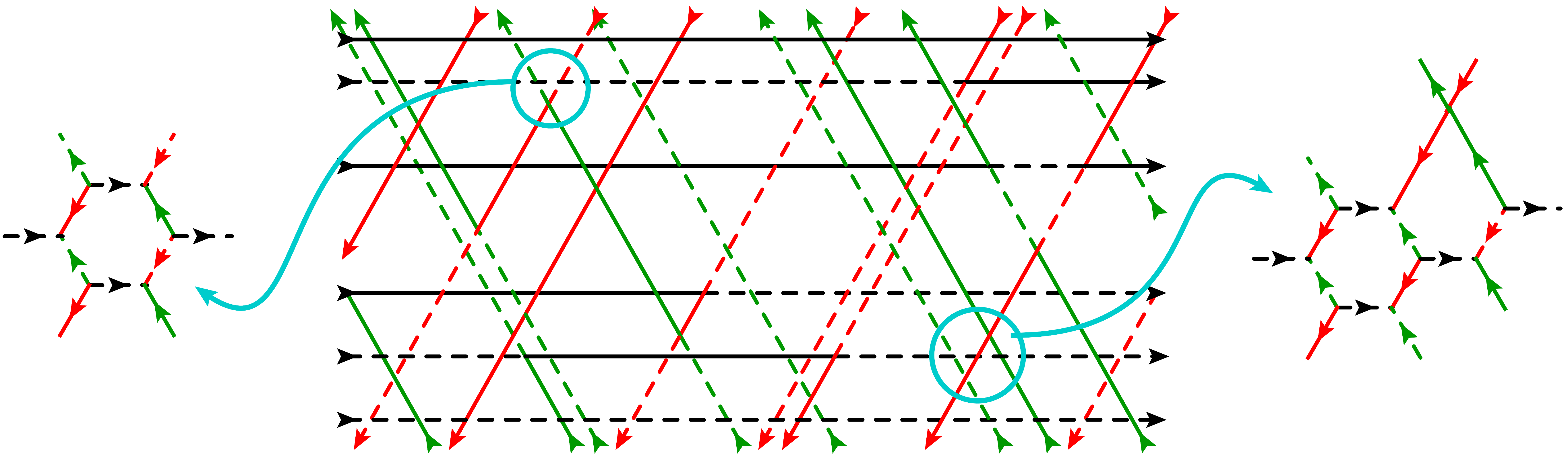
INTEGRABILITY!

[A.Zamolodchikov '80]

What about fermonic propagators?

The previous structures can be generalised considering fermionic propagators



Dynamical fishnet in D=4

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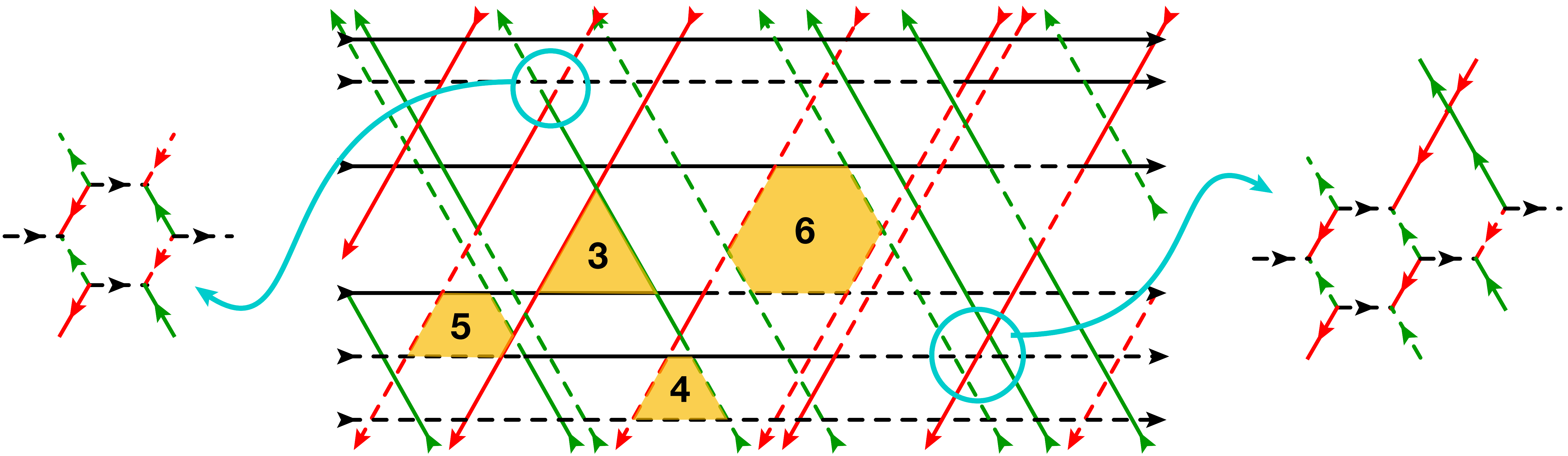

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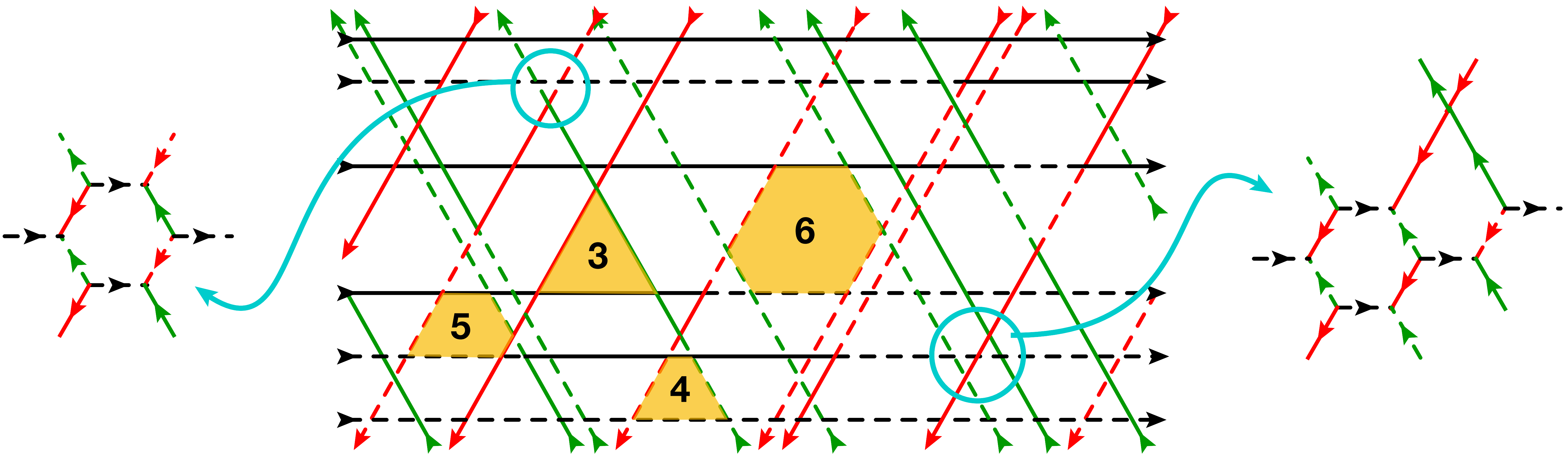

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Is there a systematic way to obtain fishnet structures from theories that we like?

γ-deformed N=4 SYM

[S.Frolov '15]
[N.Beiser, R.Roiban '15]

Replacement of the ordinary product with an associative non-commutative $\star$-product

$$A \star B = e^{-\frac{i}{2}\epsilon_{ijk}\gamma_i J_j^A J_k^B} AB$$

[O.Lunin,J.M.Maldacena, '05]

Where γ is a twist and $J^{\Phi}_{1,2,3}$ are the R-charges associated to the field Φ

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$$\mathcal{L} = N_c \mathrm{Tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D^\mu \phi_i^\dagger D_\mu \phi^i + i\bar{\psi}_A^{\dot{\alpha}} D_{\dot{\alpha}}^{\alpha} \psi_\alpha^A\Big] + \mathcal{L}_{\mathrm{int}}$$

where $i = 1,2,3\,,\ A = 1,2,3,4\,,\ D_{\dot{\alpha}}^{\alpha} = D_\mu (\sigma^\mu)_{\dot{\alpha}}^{\alpha}$ and

$$\begin{aligned}\mathcal{L}_{\mathrm{int}} = N_c g\ \mathrm{Tr}\big[&\frac{g}{4}\{\phi_i^\dagger, \phi^i\}\{\phi_j^\dagger, \phi^j\} - g\, e^{-i\epsilon^{ijk}\gamma_k}\phi_i^\dagger \phi_j^\dagger \phi^i \phi^j \\ &- e^{-\frac{i}{2}\gamma_j^-}\bar{\psi}_j \phi^j \bar{\psi}_4 + e^{+\frac{i}{2}\gamma_j^-}\bar{\psi}_4 \phi^j \bar{\psi}_j + i\epsilon_{ijk} e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+}\psi^k \phi^i \psi^j \\ &- e^{+\frac{i}{2}\gamma_j^-}\psi_4 \phi_j^\dagger \psi_j + e^{-\frac{i}{2}\gamma_j^-}\psi_j \phi_j^\dagger \psi_4 + i\epsilon^{ijk} e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+}\bar{\psi}_k \phi_i^\dagger \bar{\psi}_j\big]\end{aligned}$$

We use the notation $\gamma_1^\pm = -\frac{\gamma_3 \pm \gamma_2}{2},\ \ \gamma_2^\pm = -\frac{\gamma_1 \pm \gamma_3}{2},\ \ \gamma_3^\pm = -\frac{\gamma_2 \pm \gamma_1}{2}$ for the twists.

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Deformation parameters $q_j = e^{-\frac{i}{2}\gamma_j}\quad j = 1,2,3$, related to the Cartan subalgebras $u(1)^3 \subset su(4) \cong so(6)$, break supersymmetry!!

The double scaling limit

DS limit = Weak coupling + Large imaginary twists → $g \to 0 \quad q_i \to \infty$

$\xi_i := 4\pi q_i g$ fixed

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$$\mathcal{L}_{\phi\psi} = N_c \mathrm{Tr}\left(-\frac{1}{2}\partial^\mu \phi_j^\dagger \partial_\mu \phi^j + i\bar{\psi}_j^{\dot{\alpha}}(\sigma^\mu)_{\dot{\alpha}}^{\alpha}\partial_\mu \psi_\alpha^j\right) + \mathcal{L}_{\rm int}$$

and only a certain class of Yukawa and 4-scalar interactions survive

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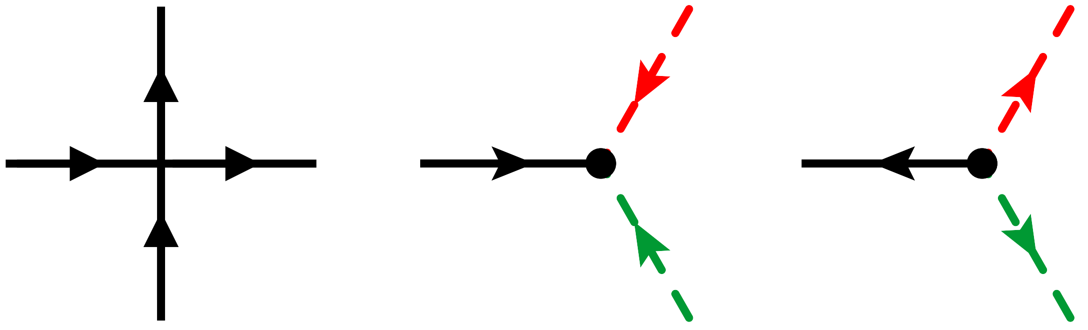
The theory is CHIRAL = Action is not invariant wrt hermitian conjugation

→ Very limited number of planar graphs.
Dynamical Fishnet



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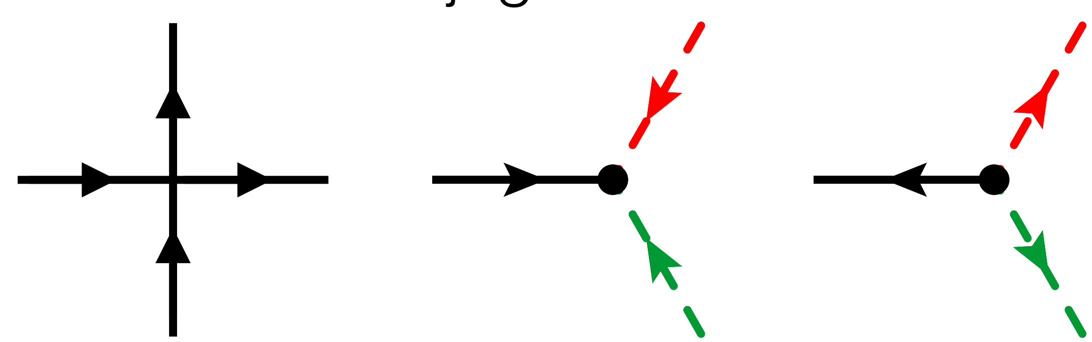

To preserve renormalisability theory must be supplemented with double traces

$$\alpha_i^2\, \mathrm{Tr}\phi_i^2\, \mathrm{Tr}\phi_i^{\dagger\,2}$$

New coupling run with the scale

$$\beta_{\alpha_i} = \frac{g^4}{\pi^2}\sin^2\gamma_i^+ \sin^2\gamma_i^- + \frac{\alpha_i^4}{4\pi^2}$$

Conformal symmetry restored imposing $\beta = 0$

A family of fishnet theories

The 3 couplings $\xi_{1,2,3}$ are free parameters that we can play with

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→ Square lattice structure!

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We are interested in studying the spectrum of anomalous dimension of those theories.

↓

This is related to Feynman periods of fishnet graphs
(see Francis/Matija review)

Spectrum: graph building operator

It is encoded in two-point functions

$$D(x) = \langle \mathcal{O}_L(x) \mathcal{O}_L^\dagger(0) \rangle = \frac{d_L(\xi)}{(x^2)^{L+\gamma_L(\xi)}} \,,$$

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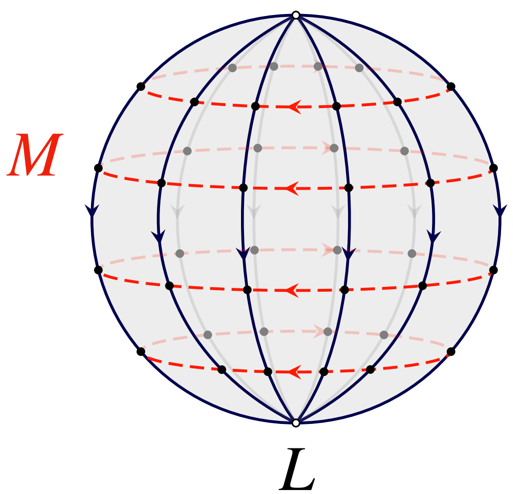
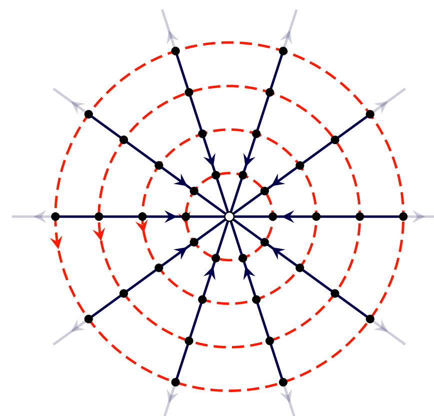


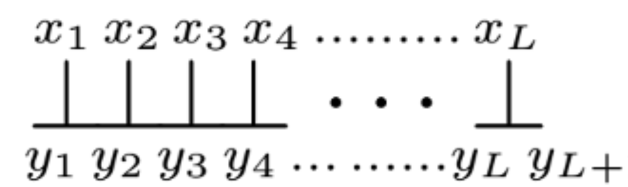


It is a fishnet graph with boundary conditions

Hamiltonian evolution in the radial direction $(x_{L+1} = x_1)$

$$\mathcal{H}_L = \prod_{l=1}^{L} \frac{1}{(x_{l+1} - x_l)^2} \prod_{l=1}^{L} \Delta_{y_l}^{-1}$$

$=$



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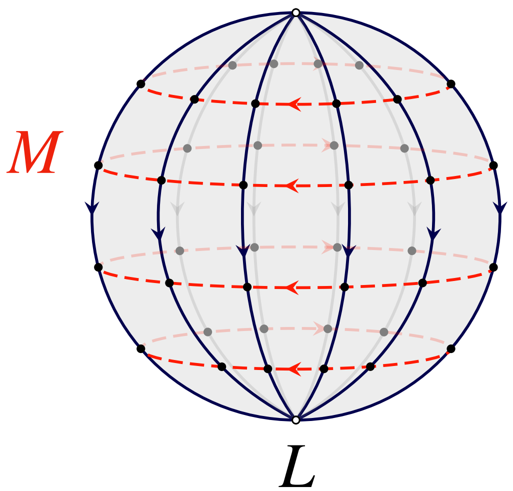

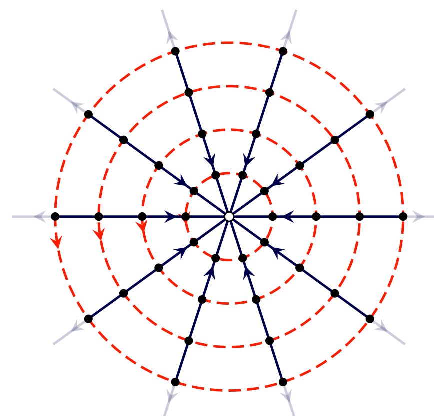

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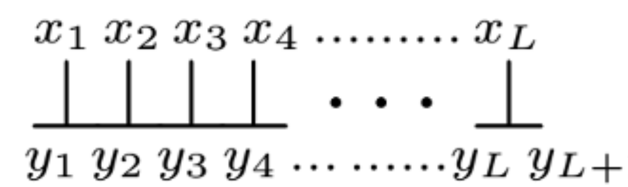
$=$



Contribution of the wheel graphs to the two-point function

$$D(x) = \sum_{M=0}^{\infty} \xi^{2LM} \int \prod_{i=1}^{L} \frac{d^4 y_i}{y_i^2} \mathcal{T}_{L,M}(x_1,\ldots,x_L|y_1,\ldots,y_L)\Big|_{x_1=\cdots=x_L=x} \,,$$

Cylinder fishnet with M wheels and L external points

Transfer matrix of a non-compact Heisenberg spin-chain

$$\mathcal{T}_{L,M} = \mathcal{H}_L \circ \mathcal{H}_L \circ \cdots \circ \mathcal{H}_L \equiv (\mathcal{H}_L)^M$$

Transfer matrix

Define R-operator acting on the tensor product of 2 reps of the conformal group

$$R_{12}(u)\,\Phi(x_1,x_2) = \int d^4y_1 d^4y_2\, R_u(x_1,x_2|y_1,y_2)\Phi(y_1,y_2)$$

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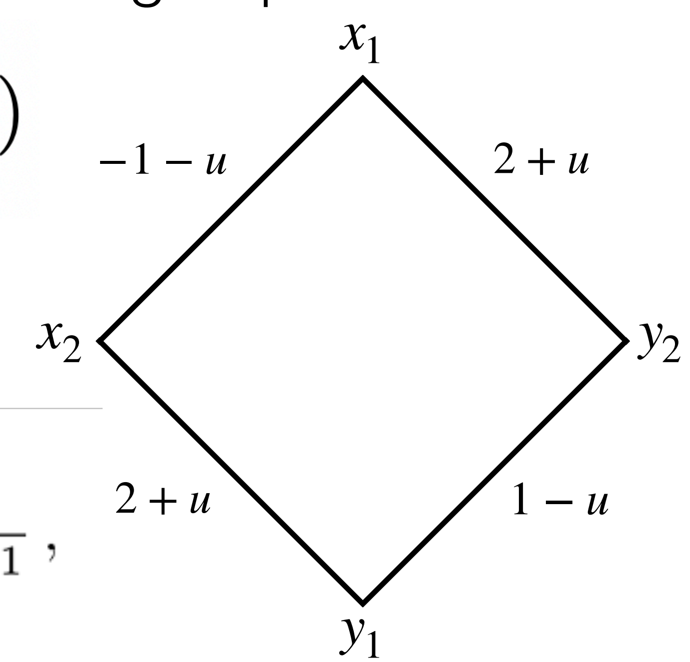
Requiring that satisfy
Yang-Baxter equation

$$R_u(x_1,x_2|y_1,y_2) = \frac{c(u)}{[(x_1-x_2)^2]^{-u-1}[(x_1-y_2)^2(x_2-y_1)^2]^{u+2}[(y_1-y_2)^2]^{-u+1}}\,,$$



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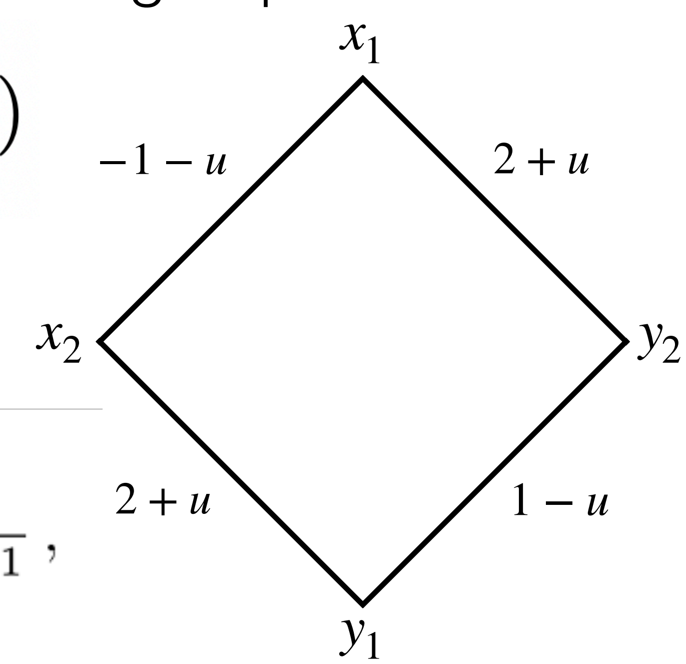

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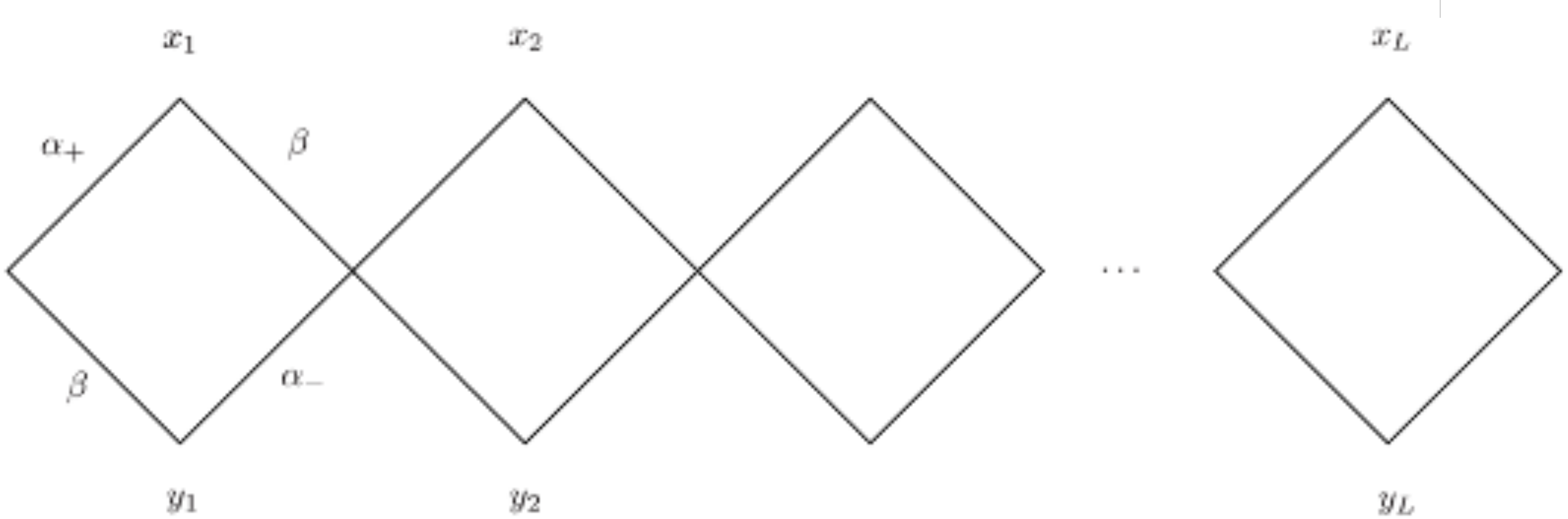
Construct the transfer matrix acting on L copies of the conformal reps

$$T_L(u) = \mathrm{tr}_0[R_{10}(u)R_{20}(u)\ldots R_{L0}(u)]$$



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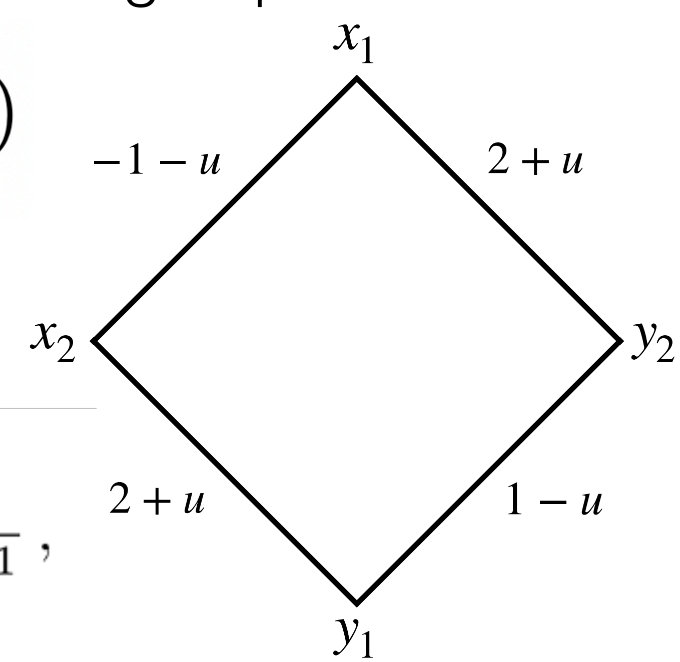

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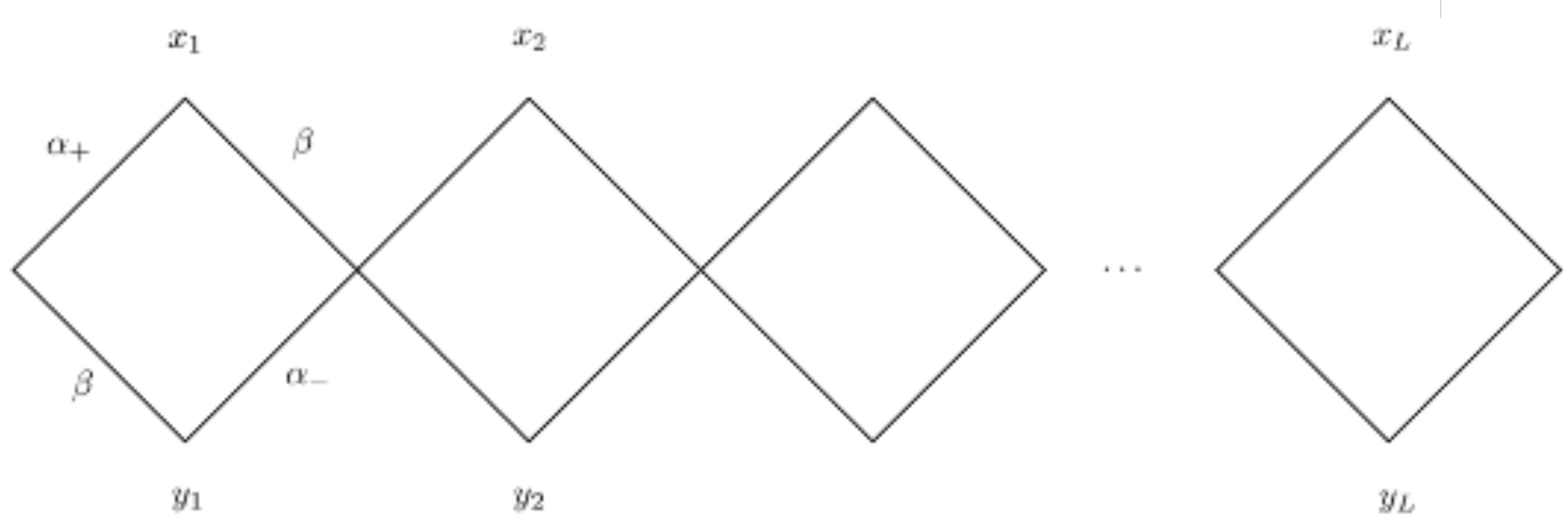

Yang-Baxter assures that transfer matrix commutes with all the integral of motion

$$[T_L(u), T_L(v)] = 0$$

INTEGRABILITY!

Graph building integrability

Transfer matrix at special value $u = -\epsilon$ with $\epsilon \to 0$ ⟶ Cyclic shift

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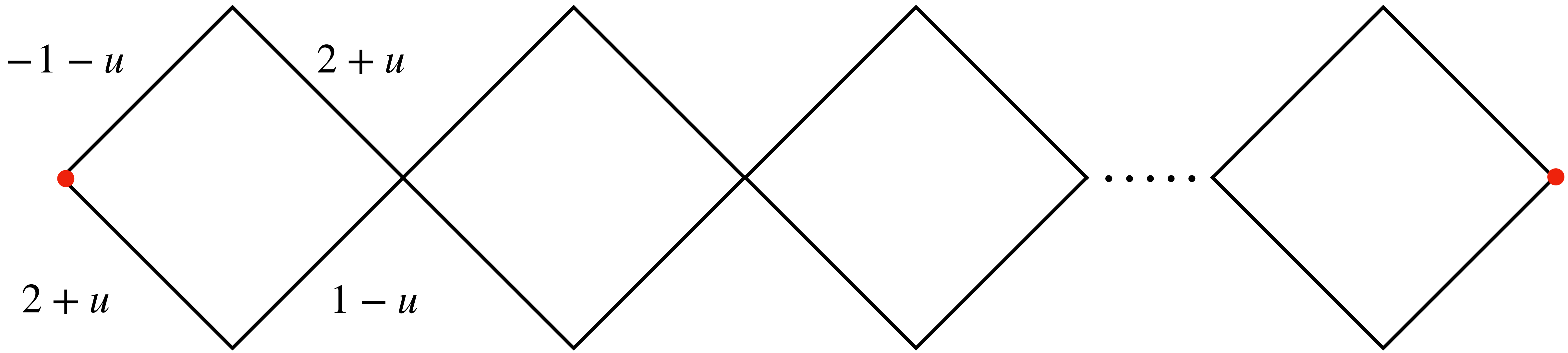
Transfer matrix at special value $u = -\epsilon$ with $\epsilon \to 0$ → Cyclic shift

Transfer matrix at special value $u = -1 + \epsilon$ with $\epsilon \to 0$



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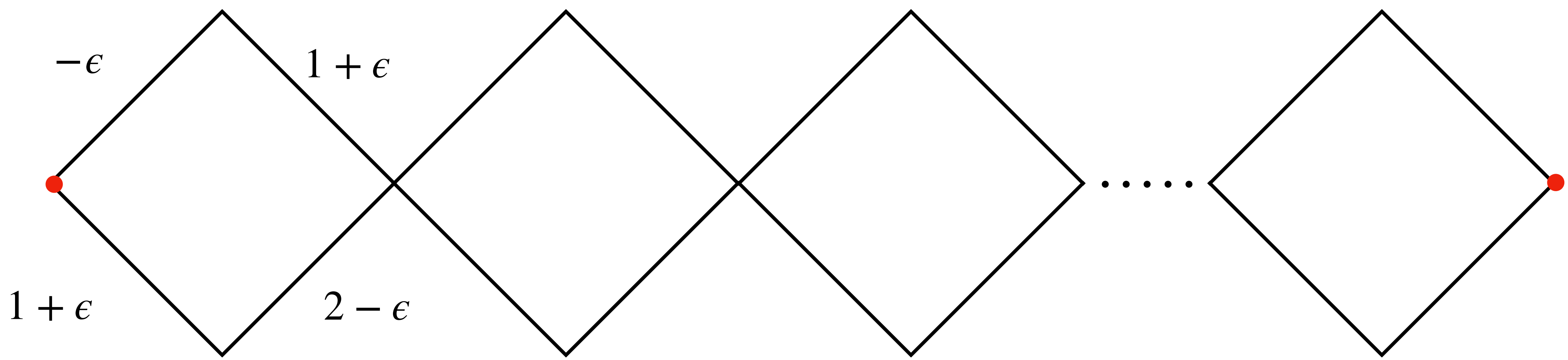

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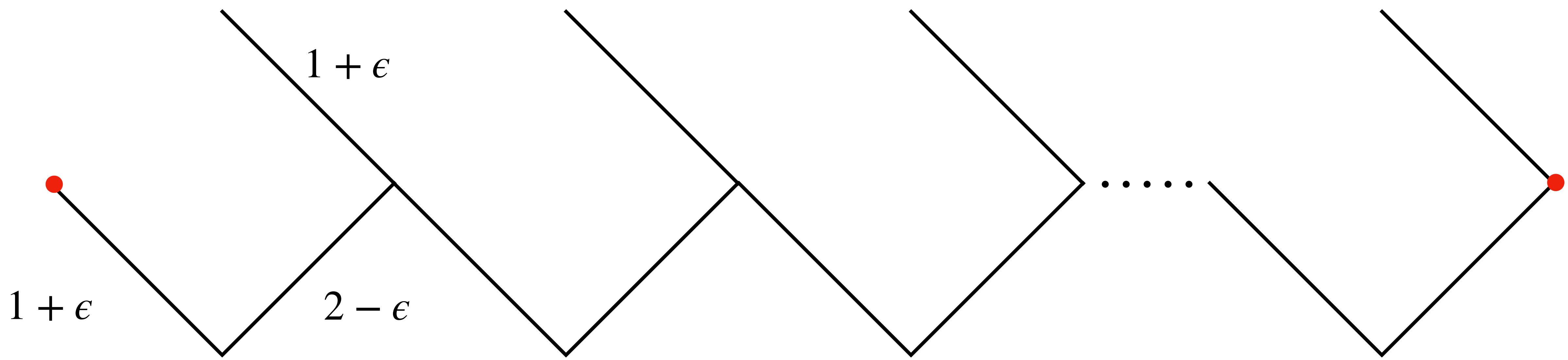

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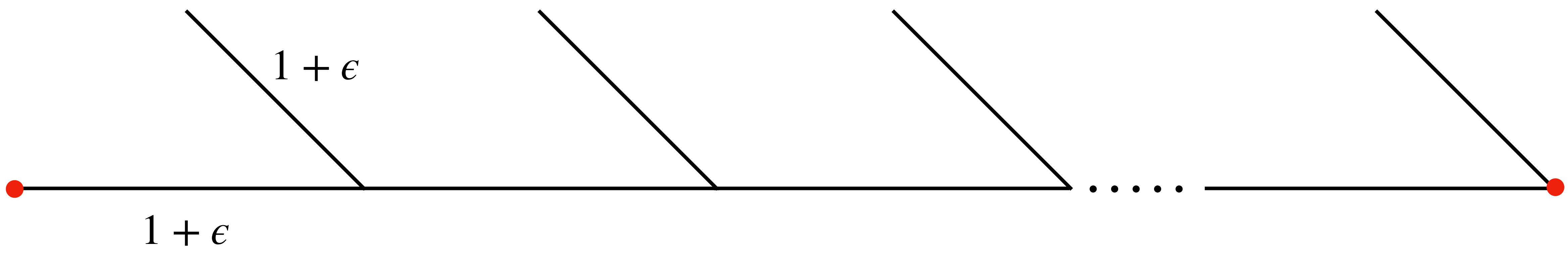

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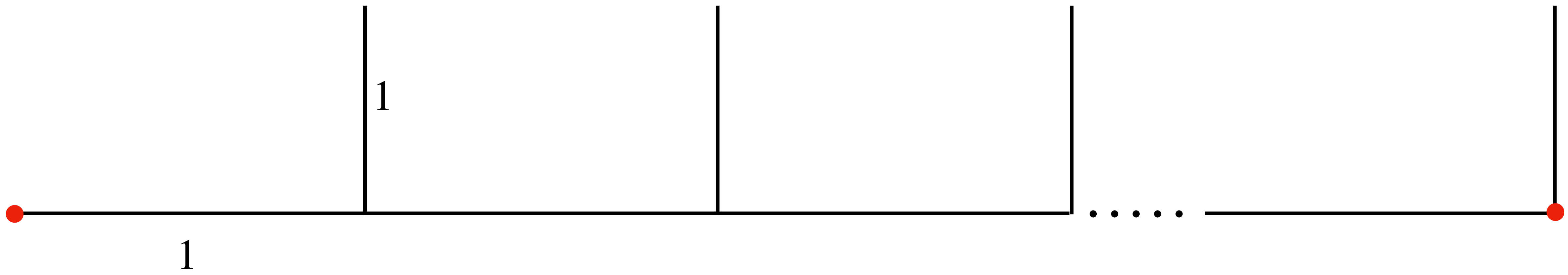

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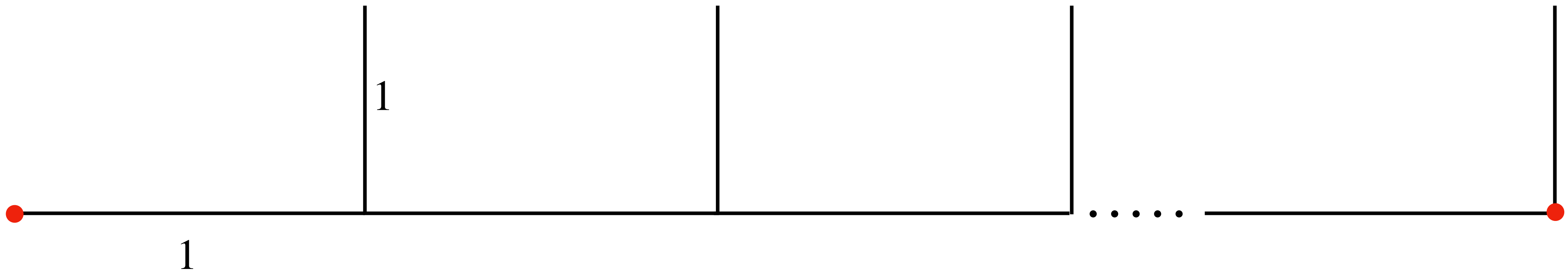

Graph-building operator is the leading asymptotic of the transfer matrix

$$T_J(-1+\epsilon) = \frac{1}{(4\epsilon)^J} \frac{1}{\Box_1 \ldots \Box_J} \prod_{i=1}^{J} \frac{1}{(x_i - x_{i+1})^2} \sim \frac{1}{\epsilon^J} \mathcal{H}_J$$

Then the graph-building operator commutes with the integral of motion

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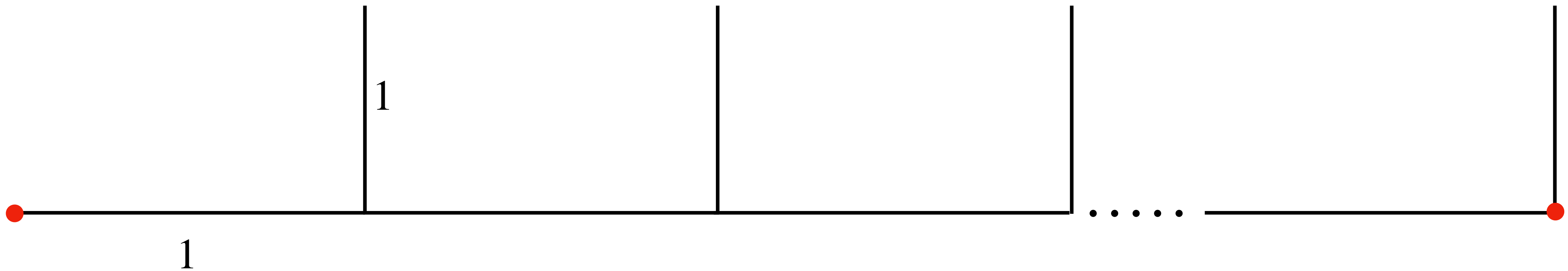

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Then the graph-building operator commutes with the integral of motion

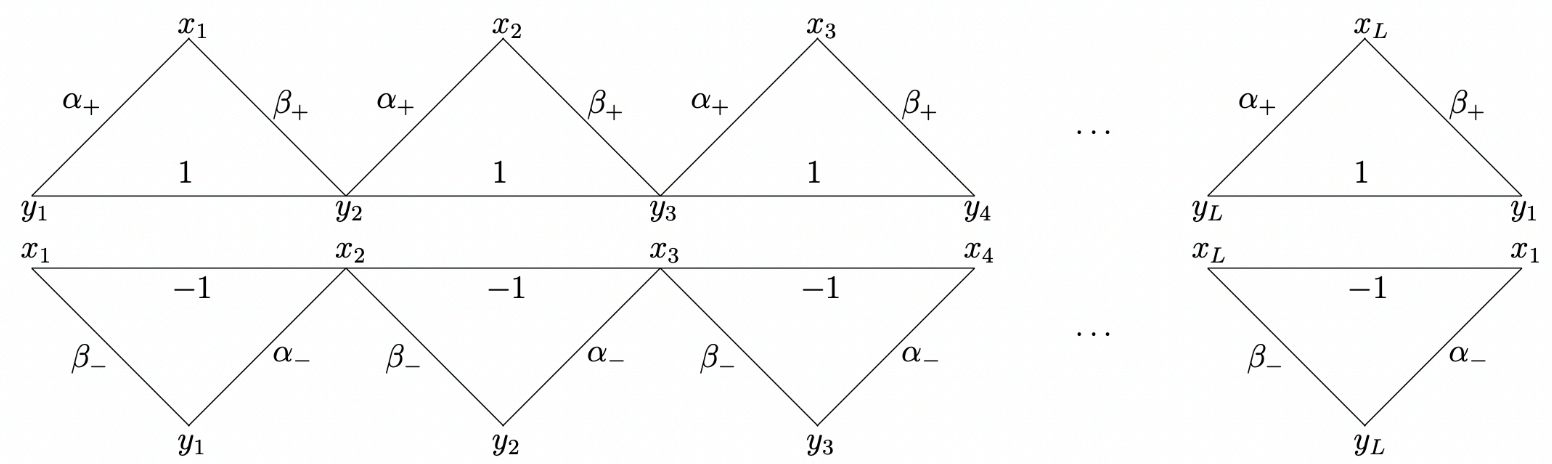
To compute the spectrum, we need to diagonalise $\mathcal{H}_J$ and to resum all eigenstates

Q-functions from field theory

$$Q_{\pm}(u)\Phi(\mathbf{x}) = \int \prod_{i=1}^{L} d^4 y_i Q_{u,\pm}(\mathbf{x}|\mathbf{y})\Phi(\mathbf{y})$$

$$Q_{u,+}(\mathbf{x}|\mathbf{y}) = \prod_{i=1}^{L} \frac{1}{|x_i - y_i|^{2\alpha_+} |x_i - y_{i+1}|^{2\beta_+} |y_i - y_{i+1}|^{2\gamma_+}},$$

$$Q_{u,-}(\mathbf{x}|\mathbf{y}) = \prod_{i=1}^{L} \frac{1}{|y_i - x_i|^{2\beta_-} |y_i - x_{i+1}|^{2\alpha_-} |x_i - x_{i+1}|^{2\gamma_-}}.$$



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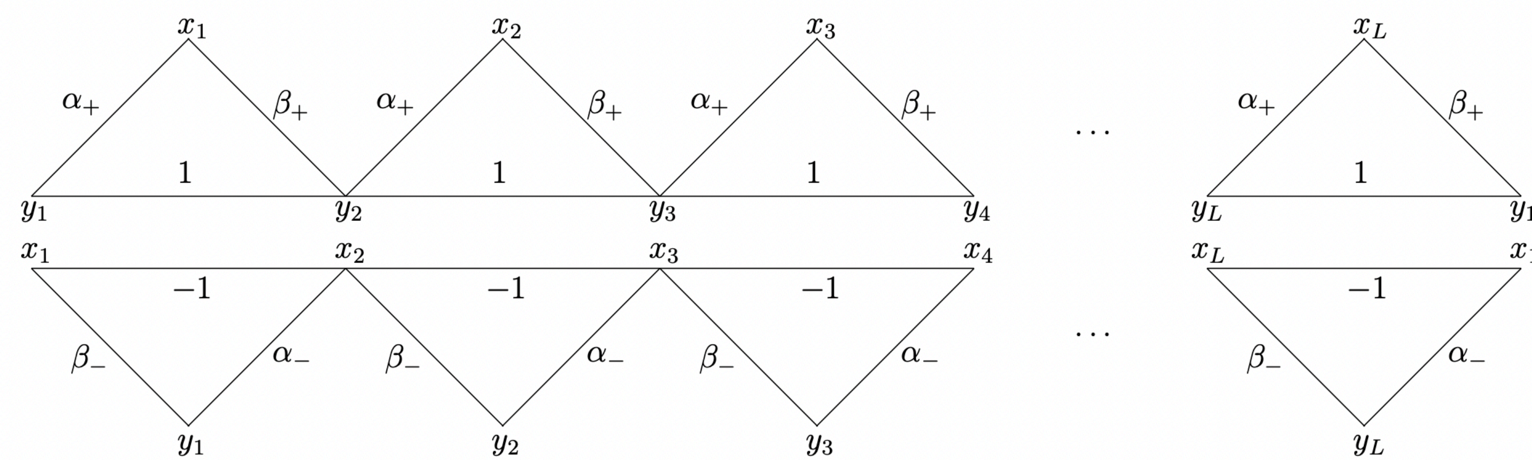
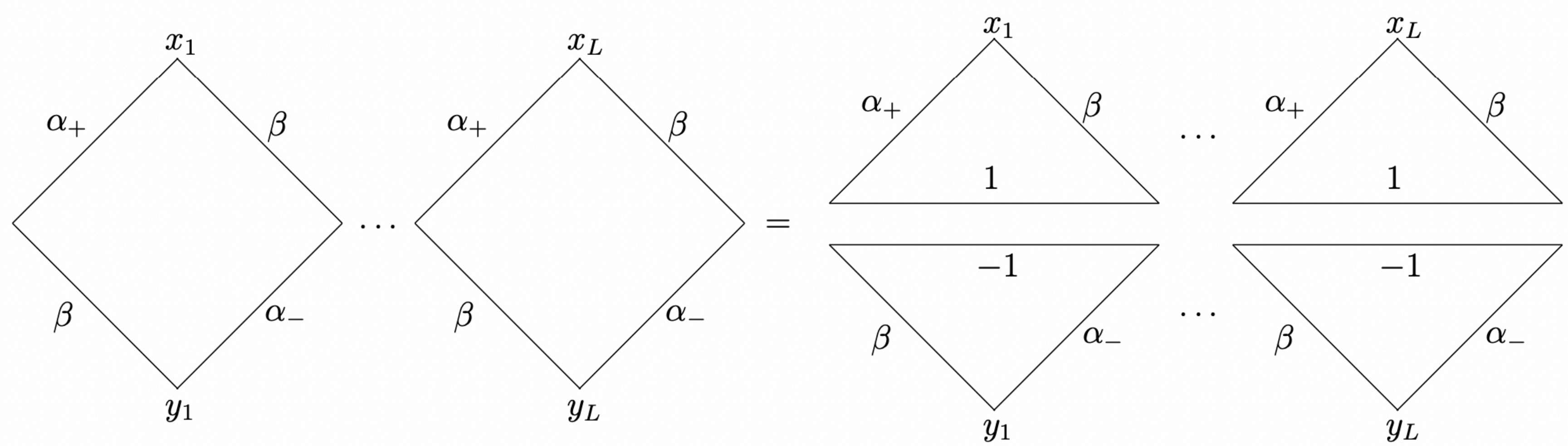


Transfer matrix can be rewritten in terms of the new integral operators



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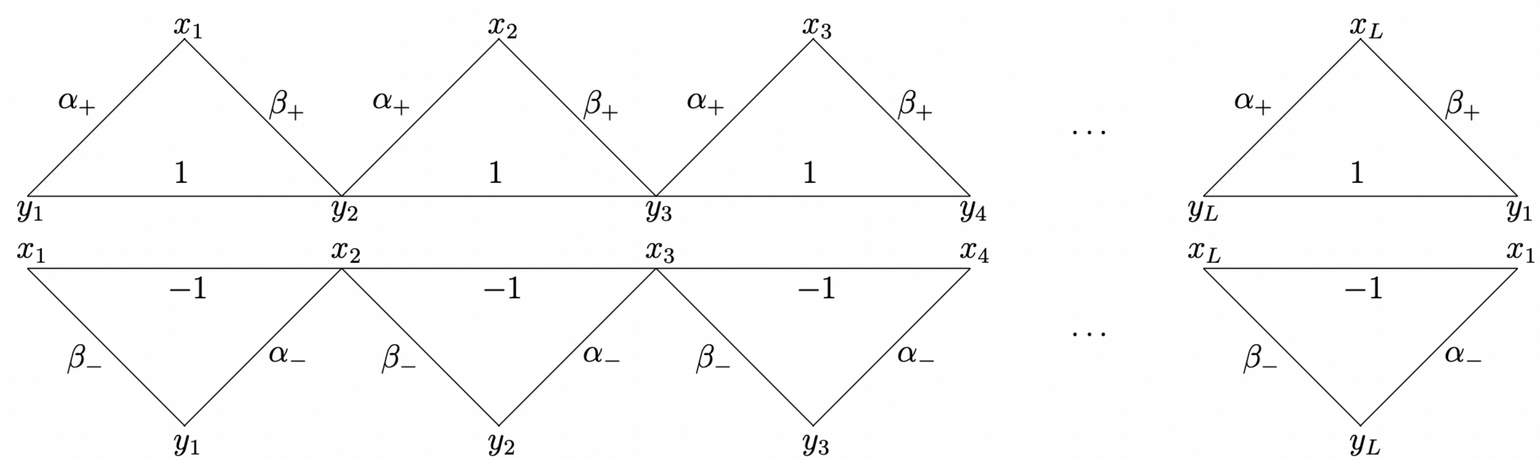

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$$T_L(u) = [c(u)]^L Q_+\left(\frac{3}{2} - u\right) Q_-\left(\frac{1}{2} - u\right)$$

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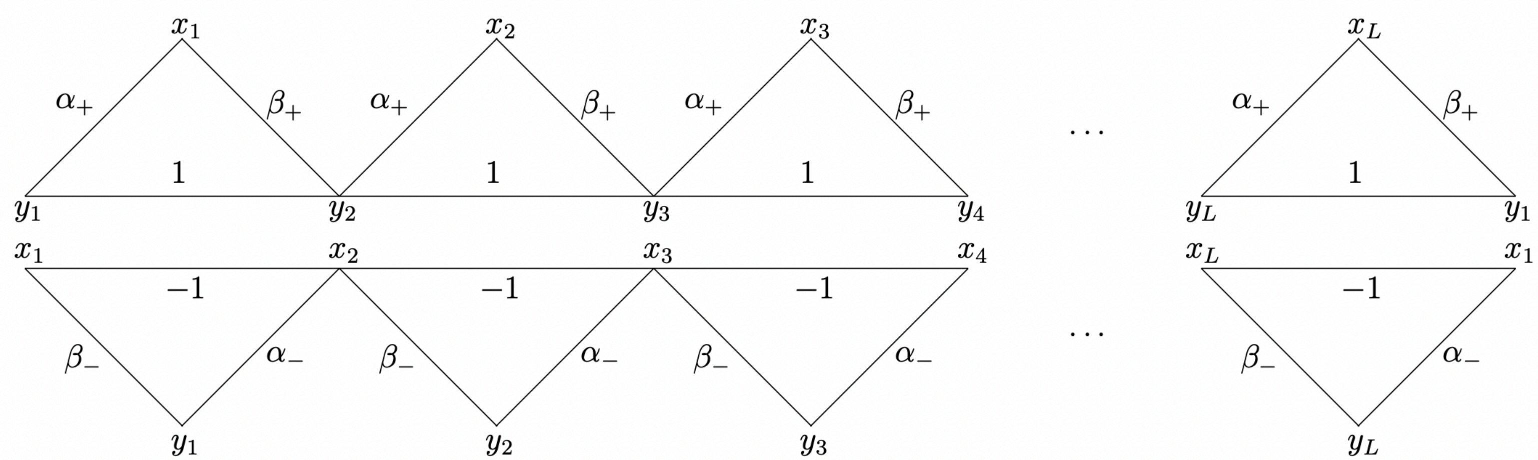

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The ratio of the transfer matrix in the limit $u = -1 + \epsilon$ and $u = -\epsilon$ with $\epsilon \to 0$,

$$\xi^{2L} = \lim_{\epsilon \to 0} \epsilon^L \frac{Q_+\left(\frac{3}{2} - \epsilon\right)}{Q_+\left(\frac{1}{2} - \epsilon\right)}$$

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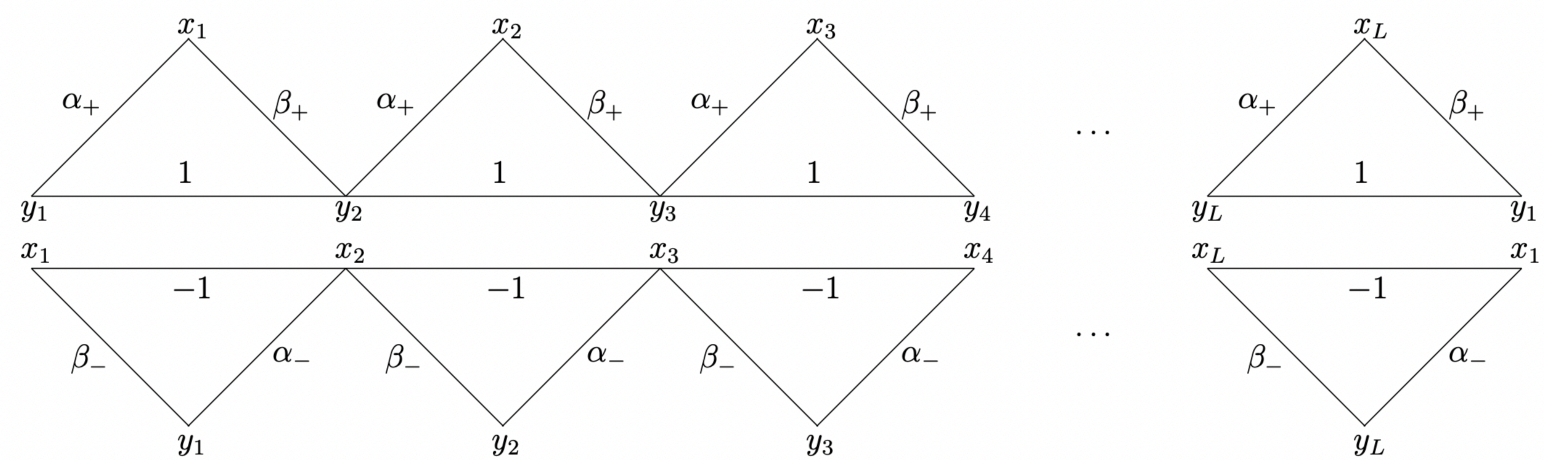

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Relates Q-functions with the coupling

They can be constructed in terms of the QSC Q-functions!!

Computing the conformal data

Consider a 4-point function of operators in the DS limit

$$\langle \mathrm{Tr}[\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)]\mathrm{Tr}[\mathcal{O}_1^\dagger(x_3)\mathcal{O}_2^\dagger(x_4)]\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^2 x_{34}^2} = \frac{\sum_{\Delta,S} C_{\Delta,S}^2 u^{(\Delta-S)/2} g_{\Delta,S}(u,v)}{x_{12}^2 x_{34}^2}$$

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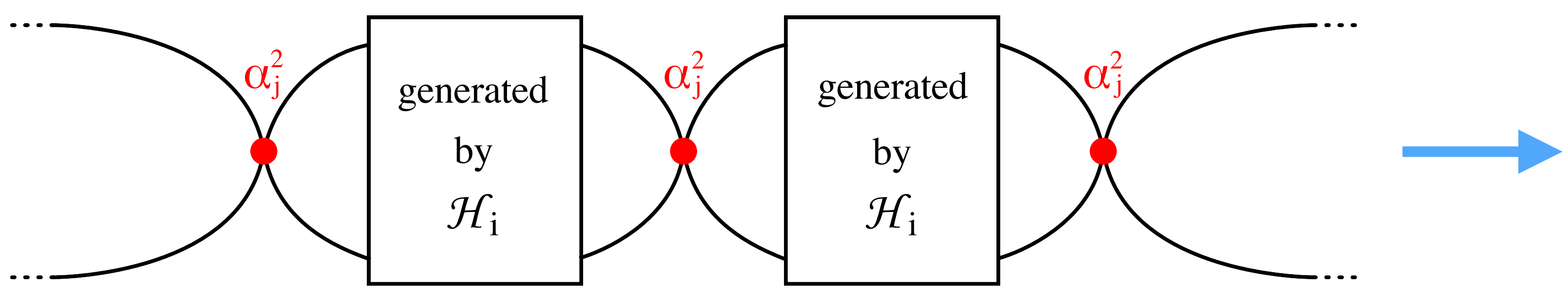
Its perturbative expansion has an iterative form that can be written as a geometric sum of primitive divergencies



$$\sum_{\ell=0}^{\infty} F[\mathcal{H}_i, \alpha_j, \xi_j]^\ell \frac{1}{x_{12}^2 x_{34}^2}$$

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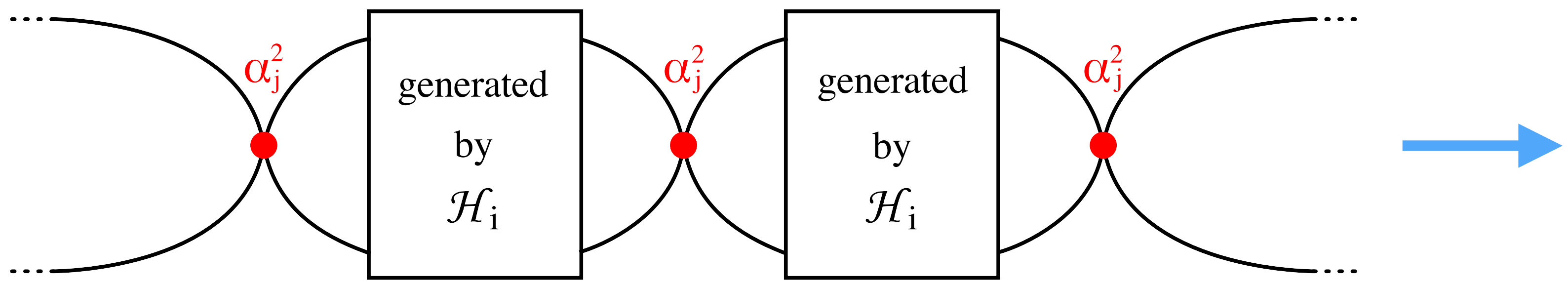

$$\sum_{\ell=0}^{\infty} F[\mathcal{H}_i, \alpha_j, \xi_j]^\ell \frac{1}{x_{12}^2 x_{34}^2}$$

The integral kernels $\mathcal{H}$ commute with the (1,0,0)x(1,0,0) spin-chain (generators of the conformal group). This property fixes their eigenstates $\Phi_{\Delta,S}$ such that

$$F[\mathcal{H}_i, \alpha_j, \xi_j]\Phi_{\Delta,S} = f[h_i(\Delta,S),\xi_j]\Phi_{\Delta,S} \quad \text{and} \quad \mathcal{G}(u,v) = \sum_S \int d\nu \mu_{\Delta,S} \frac{u^{(\Delta-S)/2} g_{\Delta,S}(u,v)}{1 - f[h_i(\Delta,s),\xi_j]}$$

where $\Delta = 2 + 2i\nu$ and $\mu_{\Delta,S}$ is given by the eigenstates normalization.

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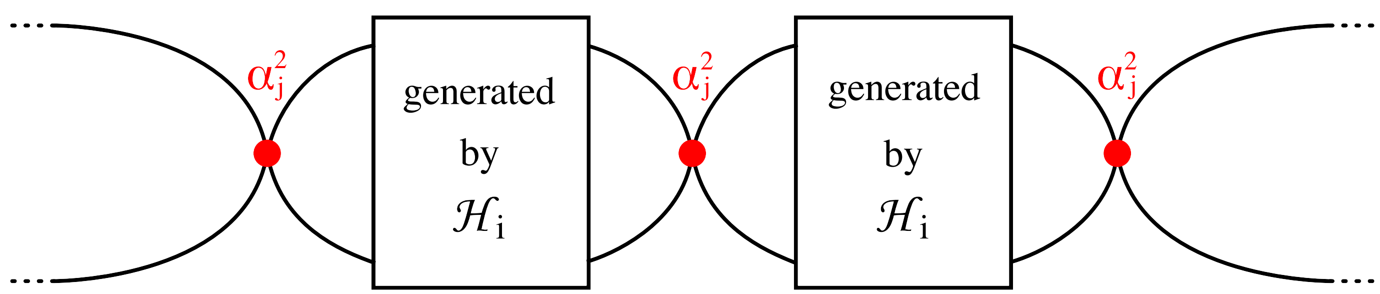

$$\rightarrow \sum_{\ell=0}^{\infty} F[\mathcal{H}_i, \alpha_j, \xi_j]^\ell \frac{1}{x_{12}^2 x_{34}^2}$$

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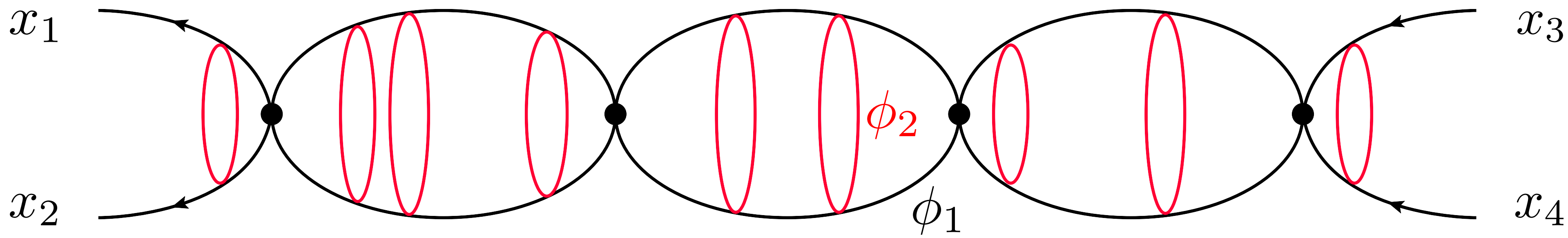
where $\Delta = 2 + 2i\nu$ and $\mu_{\Delta,S}$ is given by the eigenstates normalization.

The the conformal data can be computed using

$$C_{\Delta,S}^2 \propto \mathrm{Res}\left(\frac{d\Delta}{1 - f[h_i(\Delta,S),\xi_j]}\right), \quad \text{and} \quad \Delta \Rightarrow 1 - f[h_i(\Delta,S),\xi_j] = 0$$

Spectrum of 2-scalar theory for L=2

Consider the 4-point function $G = \langle \mathrm{Tr}[\phi_1(x_1)\phi_1(x_2)]\mathrm{Tr}[\phi_1^\dagger(x_3)\phi_1^\dagger(x_4)]\rangle$ at spin S=0



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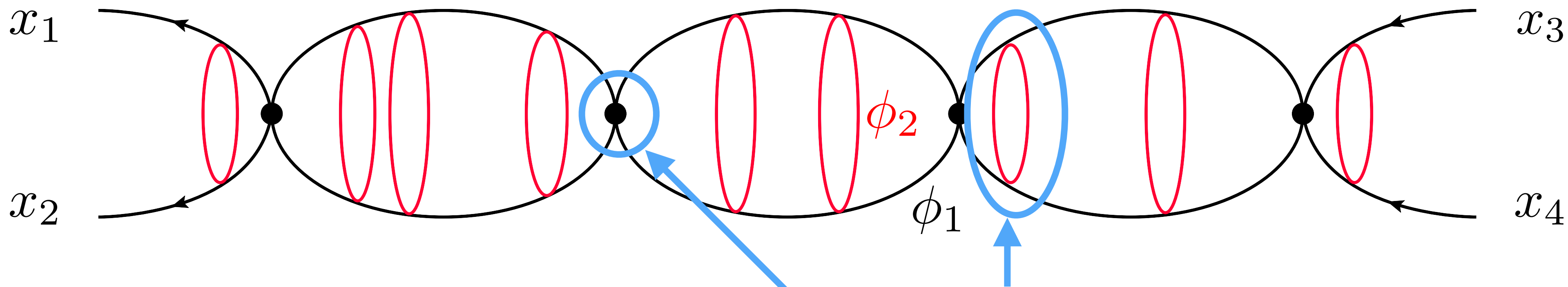

$$F = \alpha_1^2 \mathcal{V} + \xi^4 \mathcal{H}$$

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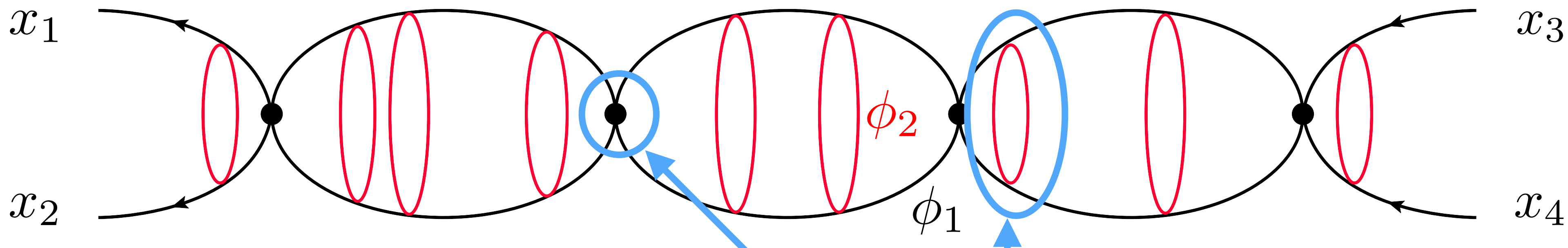
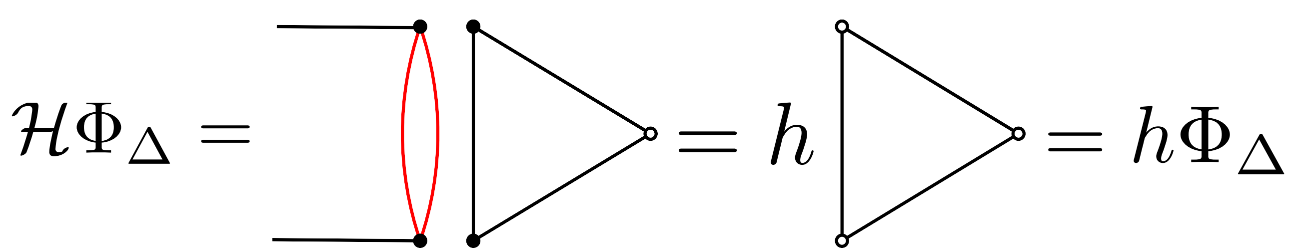
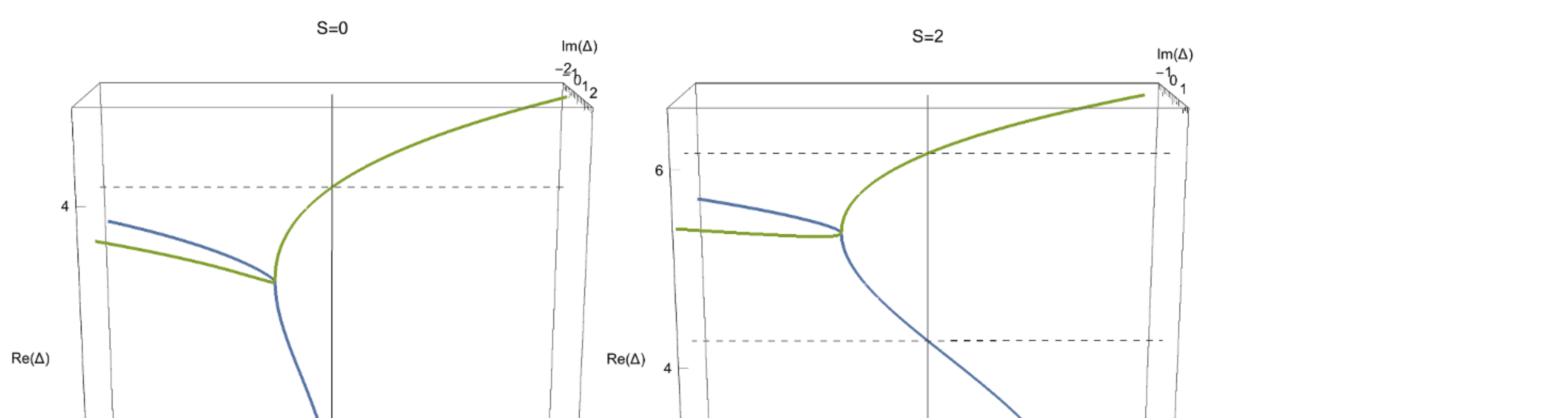


$$F = \alpha_1^2 \mathcal{V} + \xi^4 \mathcal{H}$$

To compute the scaling dimension we have to solve $h^{-1} = \xi^4$computing the eigenvalue



$$\frac{1}{16}(\Delta + S)(\Delta + S - 2)(\Delta - S - 2)(\Delta - S - 4) = \xi^4 \qquad \Delta = 2 + \sqrt{1 + (S+1)^2 \pm 2\sqrt{4\xi^4 + (S+1)^2}}$$



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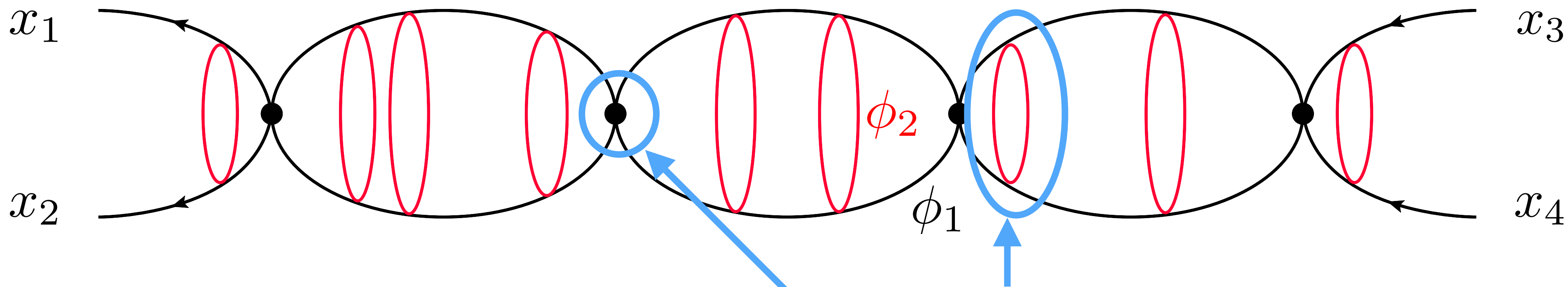

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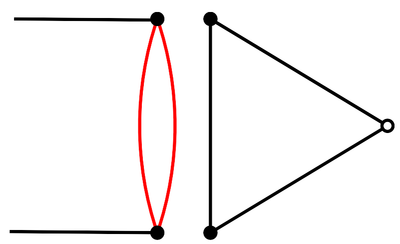
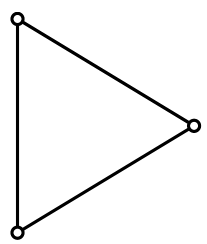
$\mathcal{H}\Phi_\Delta =$  $= h$  $= h\Phi_\Delta \;\longrightarrow\; (\Delta - 4)(\Delta - 2)^2\Delta = 16\xi^4$

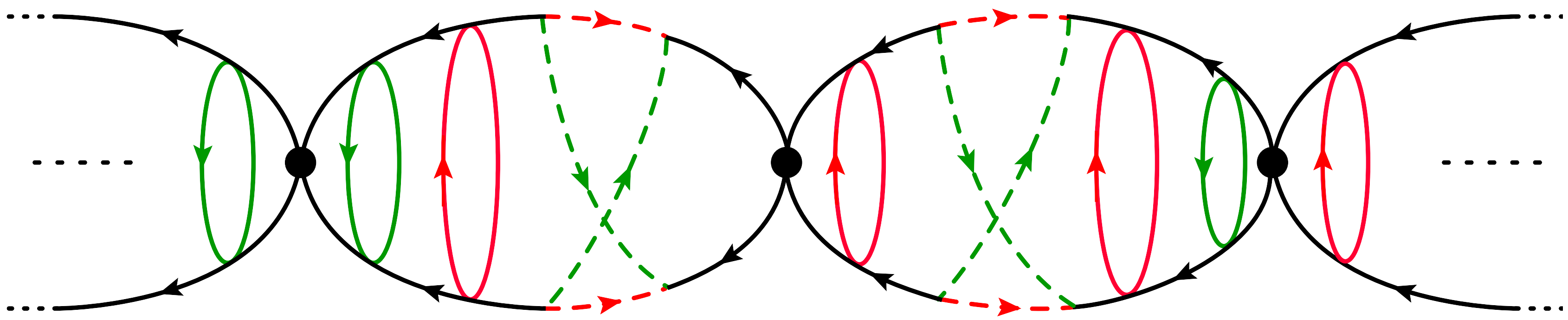
Solving the quartic equation we obtain

- twist-2 operators $\quad \Delta = 2 - 2i\xi^2 + i\xi^6 - \frac{7}{4}i\xi^{10} + \mathcal{O}(\xi^{14})$
- twist-4 operators $\quad \Delta = 4 + \xi^4 - \frac{5}{4}\xi^8 + \frac{21}{8}\xi^{12} + \mathcal{O}(\xi^{16})$
- 2 shadow operators $\quad \Delta \to 4 - \Delta$

[D.Grabner, N.Gromov, V.Kazakov, G.Korchemsky '17]

Spectrum of the DS theories for L=2

Consider the same 4-point function $G = \langle \mathrm{Tr}[\phi_1(x_1)\phi_1(x_2)]\mathrm{Tr}[\phi_1^\dagger(x_3)\phi_1^\dagger(x_4)]\rangle$



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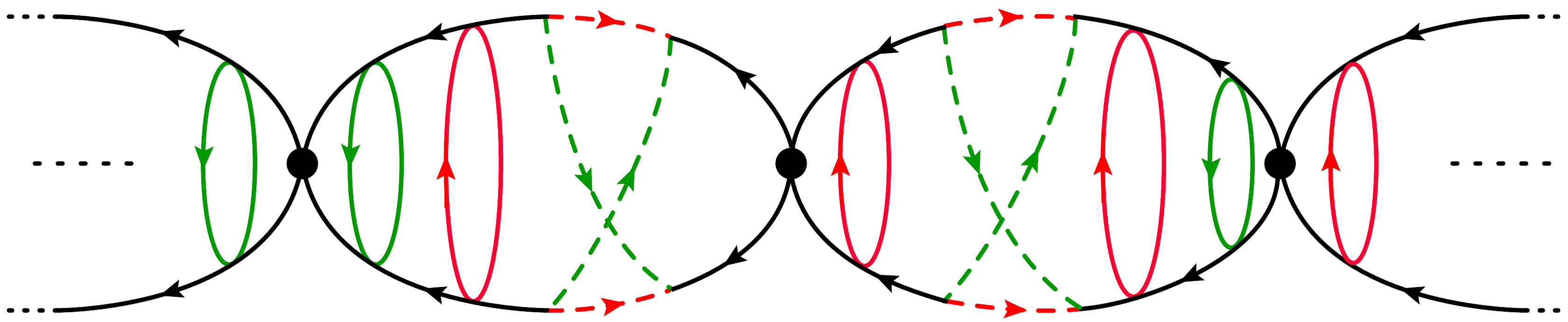
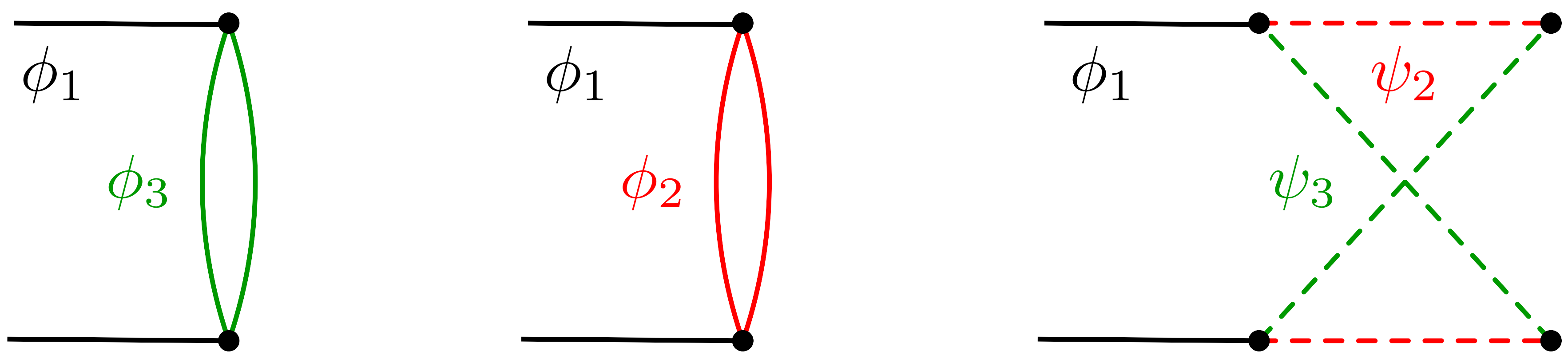


An arbitrary diagram is composed by the following bosonic and fermionic kernels



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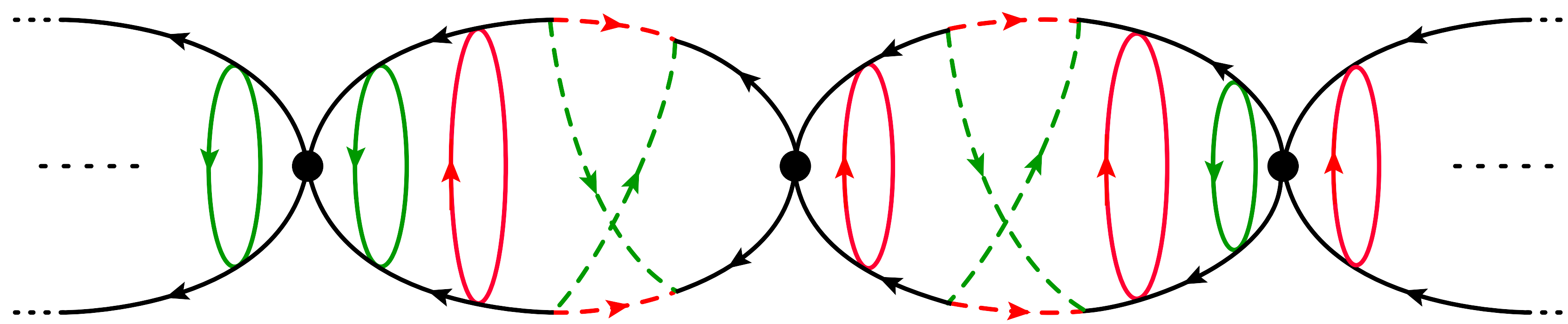

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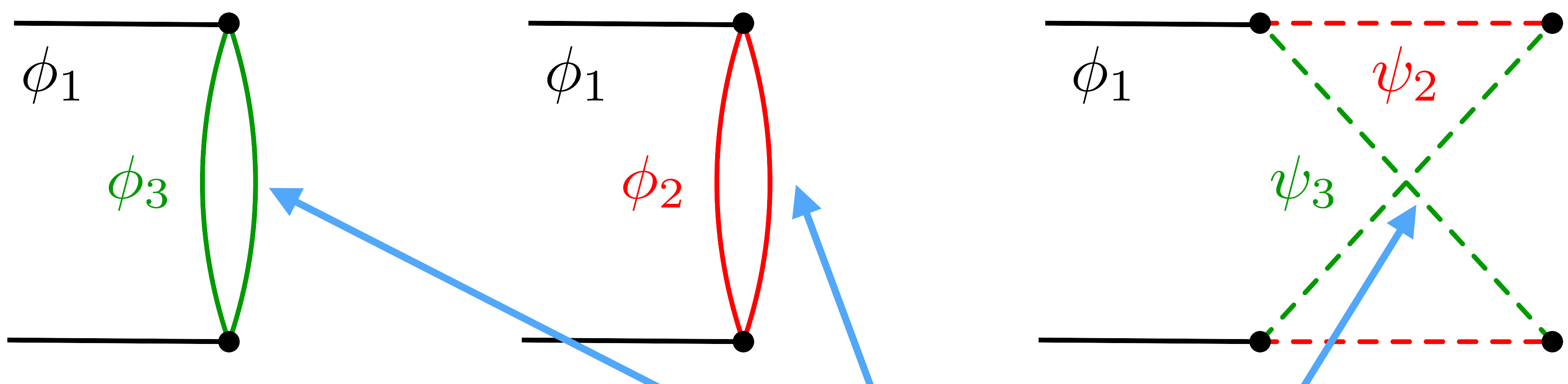

$$F = \alpha_1^2 \mathcal{V} + (\xi_2^4 + \xi_3^4)\mathcal{H} + \xi_2^2 \xi_3^2 \mathcal{H}_f$$

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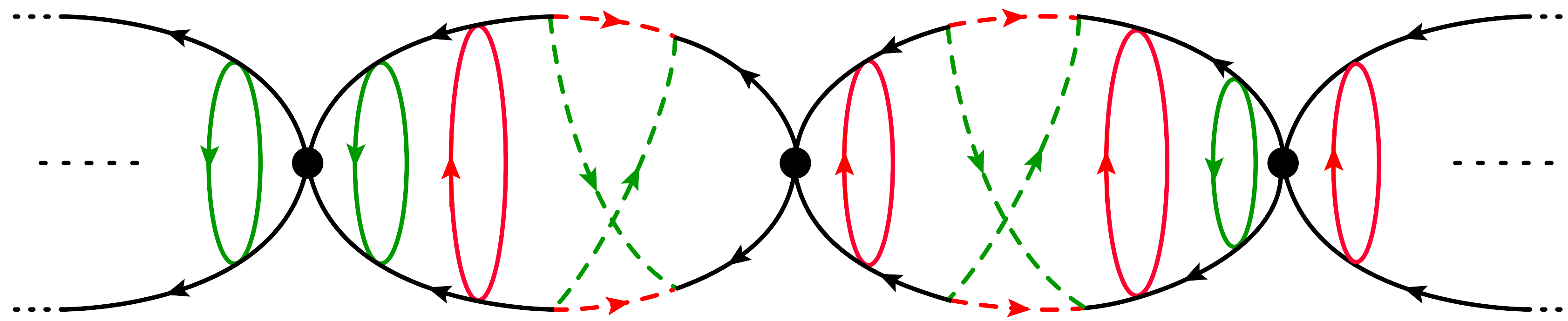

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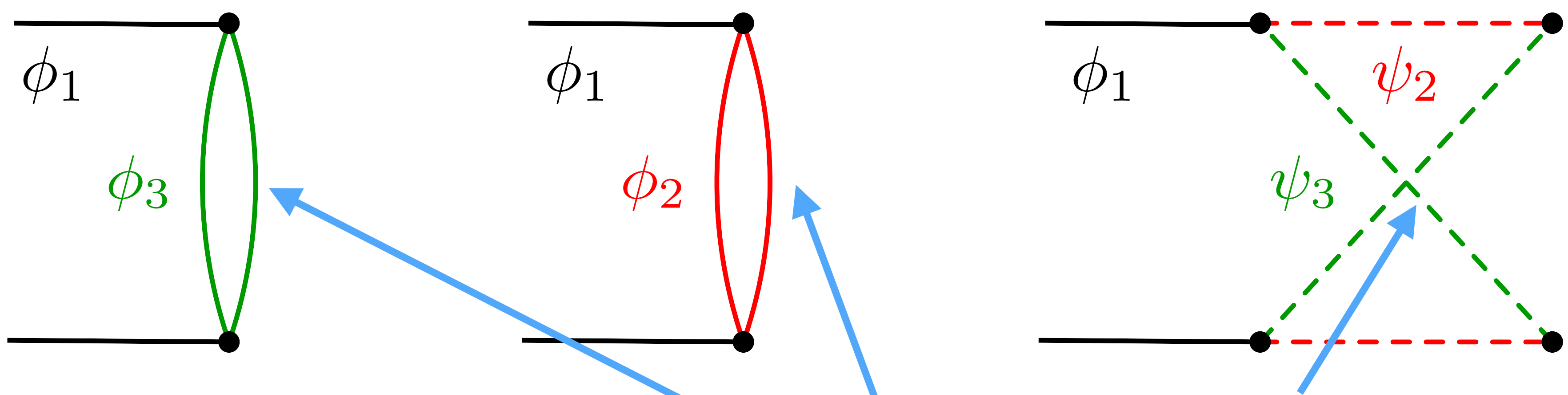

$$F = \alpha_1^2 \mathcal{V} + (\xi_2^4 + \xi_3^4)\mathcal{H} + \xi_2^2\xi_3^2\mathcal{H}_f$$

Then the spectral equation reads $h^{-1} - \xi_2^2\xi_3^2 h^{-1} h_f = \xi_2^4 + \xi_3^4$ where

$$\tilde{h}_{F\Delta,S} = \pi^4 \frac{\psi^{(1)}\left(\frac{\Delta+S}{4}\right) - \psi^{(1)}\left(\frac{\Delta+S}{4} + \frac{1}{2}\right)}{(2-\Delta)(S+1)} + (\Delta \to 4 - \Delta)$$

Spectrum of the DS theories for L=2

Let's focus only on the scaling dimension of the twist-2 operator $\mathrm{Tr}[\phi_1(x_1)\phi_1(x_2)]$

$$\Delta_1 = 2 - i\sqrt{\xi_-^2}\left[2 - \left(\xi_-^2 - 6\xi_{23}^2\zeta_3\right) + \frac{1}{4}\left(7\xi_-^4 - 12\xi_{23}^2\xi_-^2(3\zeta_3 + 5\zeta_5) + 108\xi_{23}^4\zeta_3^2\right)...\right]$$

where $\xi_- = \xi_2^2 - \xi_3^2$, and $\xi_{23} = \xi_2\xi_3$.

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Terms generated only by h_f

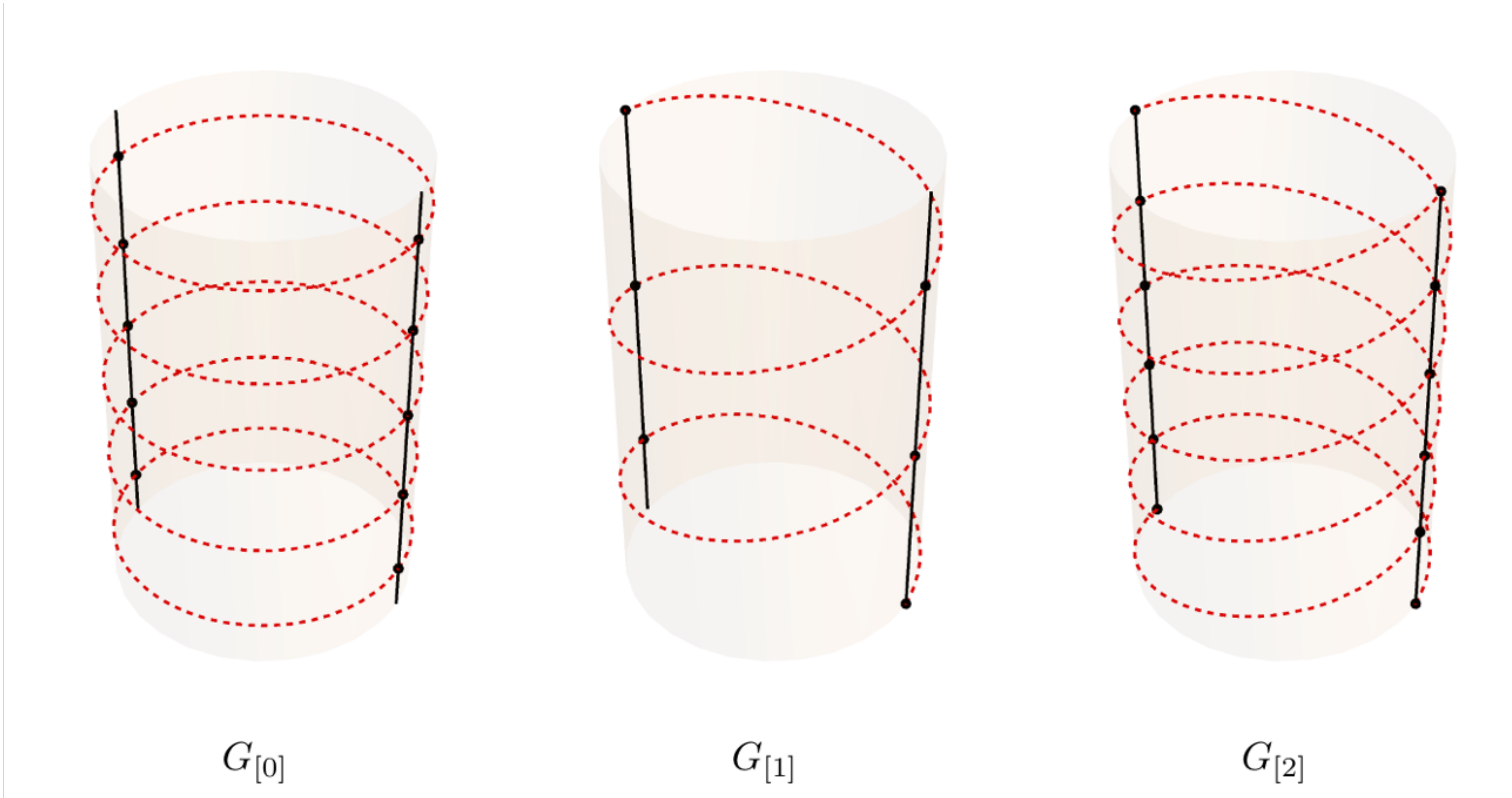
The spectrum of the other two operators with i=2,3 can be written in terms of Δ_1

$$\Delta_2 = \Delta_1(\xi_2 \to \xi_1, \xi_3) \quad \text{and} \quad \Delta_3 = \Delta_1(\xi_2, \xi_3 \to \xi_1)$$

Spectrum of the DS sub-theories? $(i, j, k = 1, 2, 3)$

- $\xi_i \to 0$ ⟶ $\Delta_i\ ,\ \Delta_{j\neq i} = \Delta^{\text{2-scalar}}$
- $\xi_i, \xi_{j\neq i} \to 0$ ⟶ $\Delta_{k\neq i,j} = 2\ ,\ \Delta_i = \Delta_j = \Delta^{\text{2-scalar}}$
- $\xi_1 = \xi_2 = \xi_3 = \xi$ ⟶ $\Delta_i = 2$ in agreement with [J.Fokken, C.Sieg, M.Wilhelm '14]

Aggiungi cosa c'e' sugli external point e che operatore generico si sta calcolando



$$\mathcal{O}_{L,n,\ell} = P_{2\ell}(\partial)\mathrm{tr}[\phi_1^L \phi_2^n (\phi_2^\dagger)^n] + \ldots\,,$$

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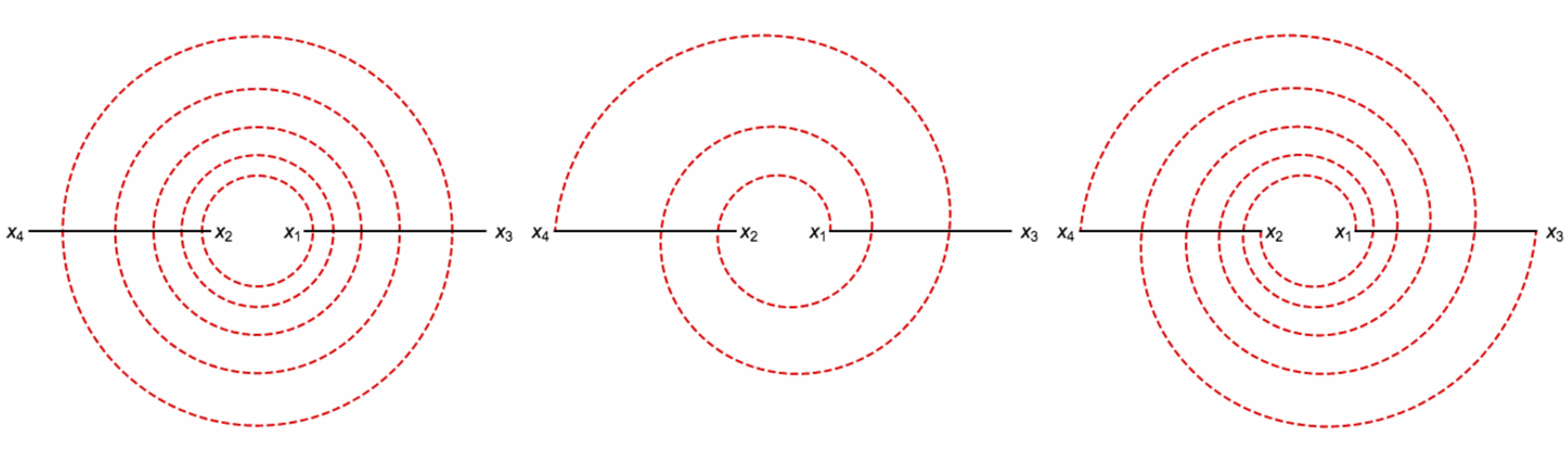


$G_{[0]}$ $G_{[1]}$ $G_{[2]}$

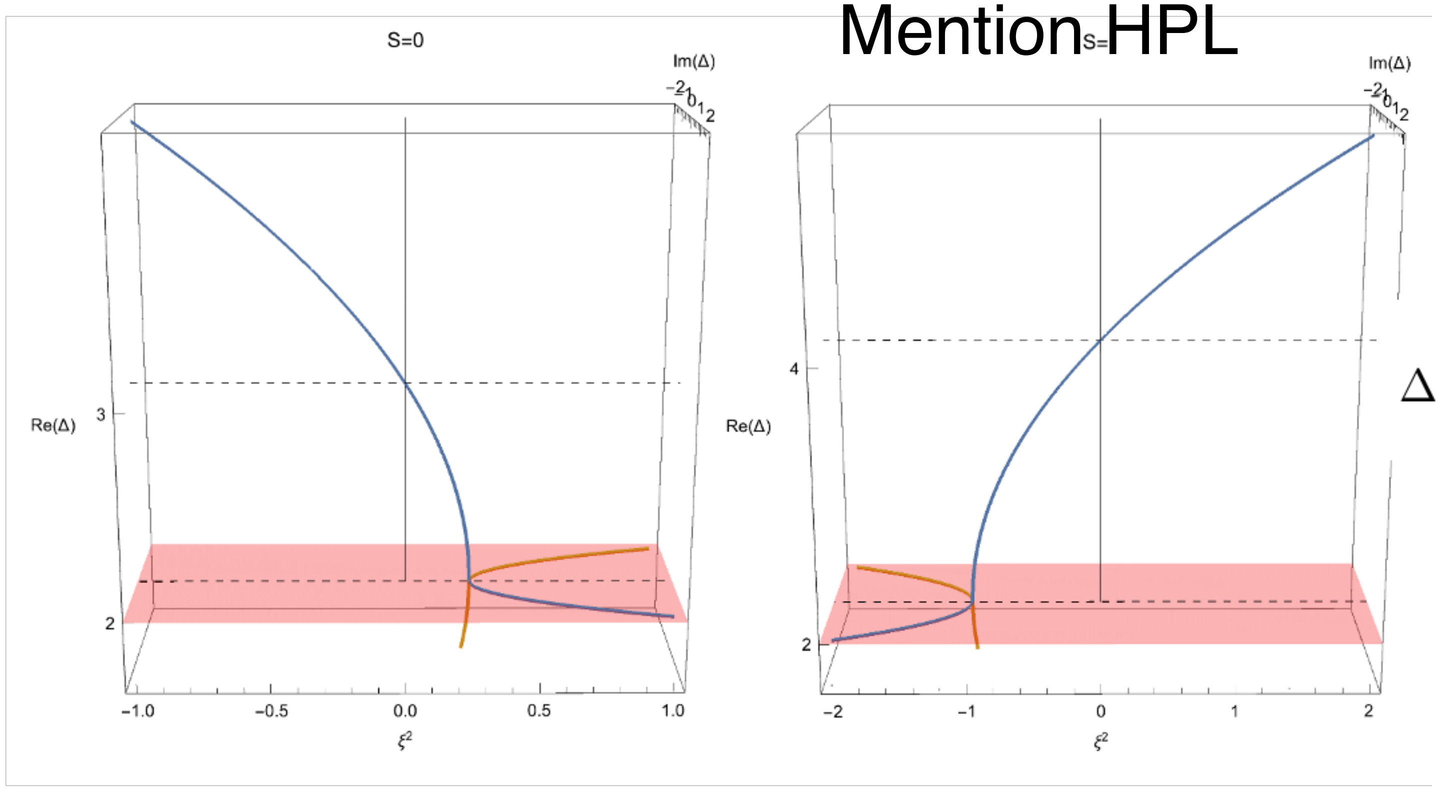
$$E_{[0]} = \frac{16\pi^4 c^4}{(-\Delta + S + 2)(-\Delta + S + 4)(\Delta + S - 2)(\Delta + S)}$$

$$E_{[2]} = \frac{\psi^{(1)}\left(\frac{1}{4}(S - \Delta + 4)\right) - \psi^{(1)}\left(\frac{1}{4}(S - \Delta + 6)\right) - \psi^{(1)}\left(\frac{S+\Delta}{4}\right) + \psi^{(1)}\left(\frac{1}{4}(S + \Delta + 2)\right)}{(16\pi^2)^2(\Delta - 2)(S + 1)}$$

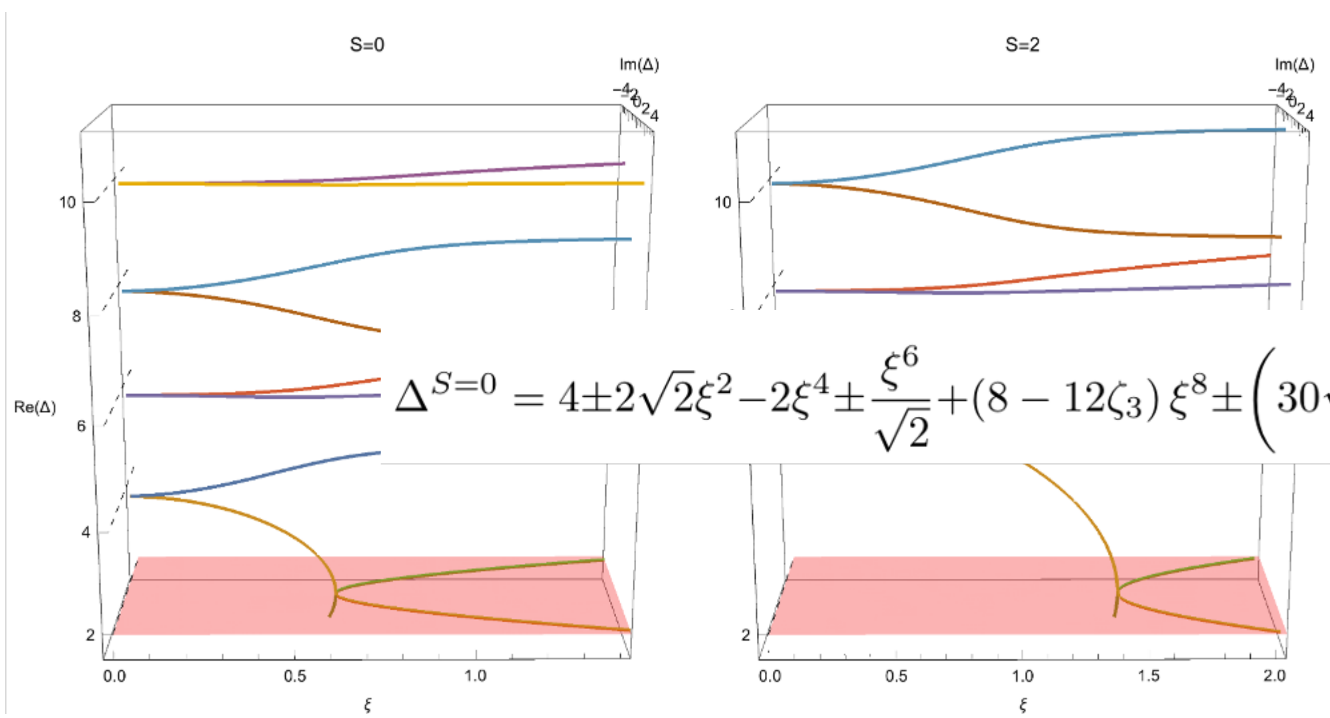
$$E_{[1]} = (-1)^S \frac{4\pi^2 c^2}{(-\Delta + S + 3)(\Delta + S - 1)}$$

$$\mathcal{O}_{L,n,\ell} = P_{2\ell}(\partial)\mathrm{tr}[\phi_1^L \phi_2^n (\phi_2^\dagger)^n] + \dots ,$$

Mention HPL



$$\Delta = 2 + \sqrt{(S+1)^2 - 4(-1)^S \xi^2}\,.$$



$$\Delta^{S=0} = 4 \pm 2\sqrt{2}\xi^2 - 2\xi^4 \pm \frac{\xi^6}{\sqrt{2}} + (8 - 12\zeta_3)\,\xi^8 \pm \left(30\sqrt{2}\zeta_3 - \frac{513}{8\sqrt{2}}\right)\xi^{10} + (-96\zeta_3 - 60\zeta_5 + 168)\,\xi^{12} \pm O\left(\xi^{14}\right)$$

Twisted QSC captures integrable deformations of N=4 such as gamma

introducing twisted boundary conditions in the spin chain picture arising at weak coupling

It is described by the same QQ-relations but Q and P have different asymptotic depending

$$\kappa_j := e^{i\,\gamma_j^+} \quad \text{and} \quad \hat{\kappa}_j := e^{i\,\gamma_j^-} \qquad j = 1, 2, 3\,,$$

$$\mathbf{P}_a \sim A_a x_a^{iu} u^{-\hat{\lambda}_a}\,, \qquad \mathbf{P}^a \sim A^a x_a^{-iu} u^{\hat{\lambda}_a^*}\,,$$

$$\mathbf{Q}_i \sim B_i u^{-\hat{\nu}_i} \left(1 + \frac{b_{i,1}}{u} + \frac{b_{i,2}}{u^2} + \dots\right)\,,$$

$$\mathbf{Q}^i \sim B^i u^{\hat{\nu}_i^*} \left(1 + \frac{b^{i,1}}{u} + \frac{b^{i,2}}{u^2} + \dots\right)\,.$$

$$= \left\{\frac{J}{2}, \frac{J}{2}, -\frac{J}{2}, -\frac{J}{2}\right\} \qquad x_a = \left\{\kappa^J, \kappa^{-J}, \hat{\kappa}^J, \hat{\kappa}^{-J}\right\}$$

$$\frac{\Delta}{2}, -1 - \ldots$$

$$B_1 B^1 = -B_4 B^4 = \frac{i(\kappa^J - 1)^2(\hat{\kappa}^J - 1)^2}{(\kappa\hat{\kappa})^J(\Delta - 2)(\Delta - 3)}$$

$$\frac{\Delta}{2} + 3, -\ldots$$

$$B_2 B^2 = -B_3 B^3 = -\frac{i(\kappa^J - 1)^2(\hat{\kappa}^J - 1)^2}{(\kappa\hat{\kappa})^J(\Delta - 1)(\Delta - 2)} \qquad \ldots \hat{\kappa})^J - 1)$$

$$A_3 = -A_4 = -\frac{\kappa^J(\hat{\kappa}^J - 1)^3}{(1 + \hat{\kappa}^J)(\kappa^J - \hat{\kappa}^J)((\kappa\hat{\kappa})^J - 1)}$$

$$\mathbf{Q}_i^{[+4]} D_0 - \mathbf{Q}_i^{[+2]} \left[D_1 - \mathbf{P}_a^{[+2]} \mathbf{P}^{a[+4]} D_0\right] + \mathbf{Q}_i \left[D_2 - \mathbf{P}_a \mathbf{P}^{a[+2]} D_1 + \mathbf{P}_a \mathbf{P}^{a[+4]} D_0\right]$$
$$-\mathbf{Q}_i^{[-2]} \left[\bar{D}_1 + \mathbf{P}_a^{[-2]} \mathbf{P}^{a[-4]} \bar{D}_0\right] + \mathbf{Q}_i^{[-4]} \bar{D}_0 = 0\,,$$

Diagonalise with baxter

$$g \to 0 \text{ and } s \equiv \sqrt{\kappa\hat{\kappa}} \to \infty \text{ with } \xi = gs \text{ fixed}$$

$$q_i(u) = \mathbf{Q}_i(u)u^{-J/2}$$

$$\left(\frac{\Delta(\Delta-2)}{4u^2} - 2\right) q(u) + q(u+i) + q(u-i) = 0$$

$$\begin{aligned} Q_+(u) = {} & q_1\left(u+\frac{i}{2}\right) q_2\left(-u-\frac{i}{2}\right) + q_1\left(-u-\frac{i}{2}\right) q_2\left(u+\frac{i}{2}\right) \\ & + A\left[q_3\left(u+\frac{i}{2}\right) q_4\left(-u-\frac{i}{2}\right) + q_3\left(-u-\frac{i}{2}\right) q_4\left(u+\frac{i}{2}\right)\right], \end{aligned}$$

Dependence of Delta on m and xi?

Quantisation fixes Delta on m

$$\left(\frac{(\Delta-1)(\Delta-3)}{4u^2}-\frac{m}{u^3}-2\right)q(u)+q(u+i)+q(u-i)=0$$

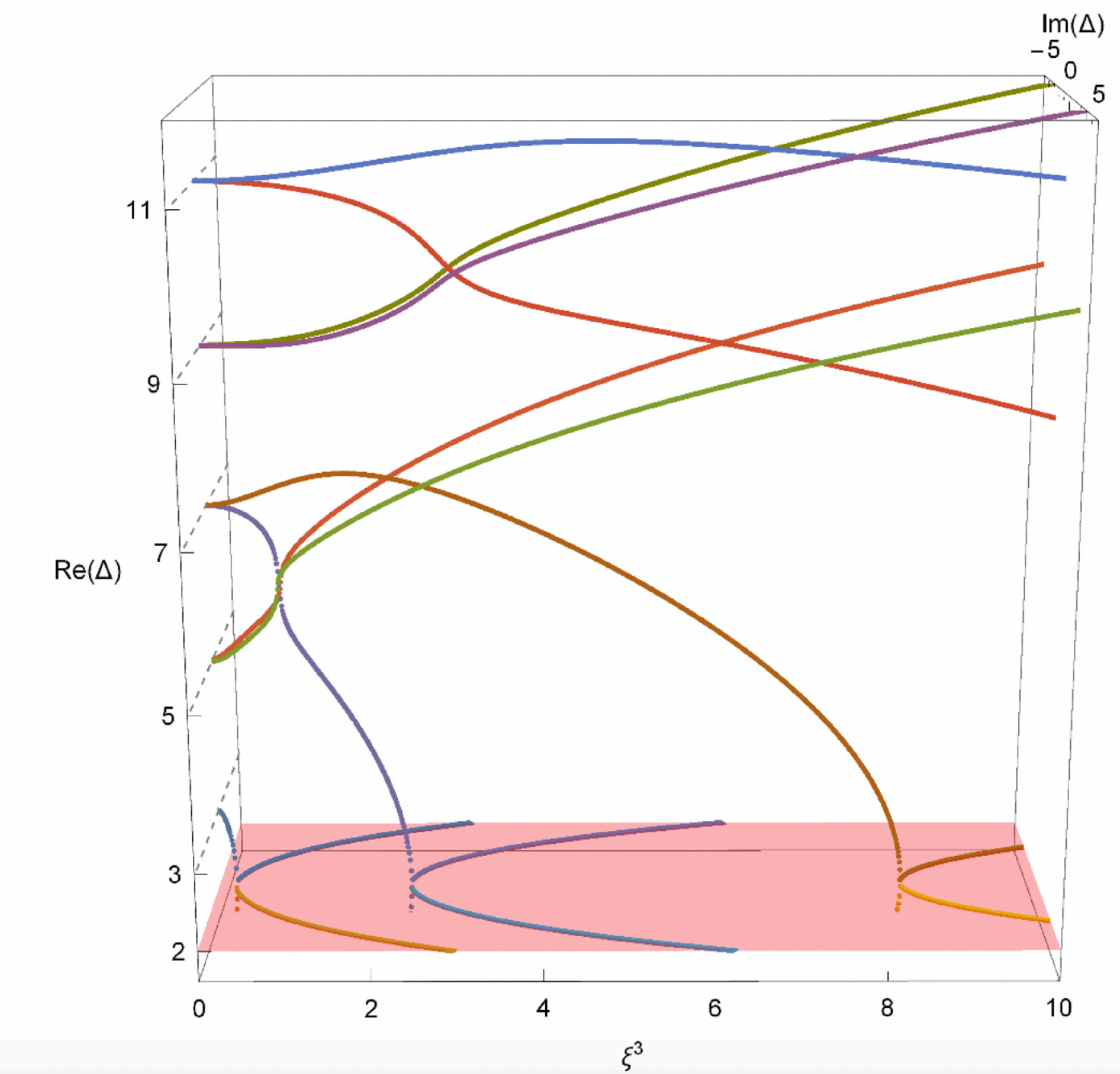
$$\tilde{\mathbf{Q}}_i(-u)=H_i{}^j\mathbf{Q}_j(u), \qquad H_i{}^j=\begin{pmatrix} 0 & 0 & 0 & -\beta \\ \frac{1}{\beta} & 0 & 0 & 0 \\ 0 & -\frac{1}{\beta} & 0 & 0 \end{pmatrix},$$

$$\mathbf{Q}_i(-u)=\Omega_i{}^j(u)\mathbf{Q}(u), \quad \Omega(u+i)=\Omega(u)$$

$$q_4(0,m)q_2(0,-m)+q_2(0,m)q_4(0,-m)=0$$

$$m^2=-g^6s^6=-\xi^6$$

Numerics



We need 2 independent solutions of the Baxter equation with $m^2 = \xi^6$
We want a perturbative solution in powers of m. Leading order
$\Delta(m = 0) = L = J + 2n$

$$q_I = \mathcal{P}_{(L-1)/2}(u)\,, \qquad q_{II} = \mathcal{P}_{(L-1)/2}(u)\eta_2(u) + \mathcal{Q}_{(L-3)/2}(u)$$

$$\eta_{s_1,\dots,s_k}(u) = \sum_{n_1 > n_2 > \dots > n_k \geq 0} \frac{1}{(u + in_1)^{s_1} \dots (u + in_k)^{s_k}}$$

$$L = 3 \;:\quad q_I = u\,, \qquad q_{II} = 1\,,$$

$$L = 5 \;:\quad q_I = u^2\,, \quad q_{II} = u^2\eta_2(u) + iu - \frac{1}{2}\,,$$

$$L = 7 \;:\quad q_I = u^3\,, \quad q_{II} = u^3\eta_2(u) + iu^2 - \frac{u}{2} - \frac{i}{6}\,.$$

How to iterate to next order

Quantisation +result for L=3

Higher L matching numerics

L=2 gamma deformed

Method

Analytic data

Limit for fishnet