Coulomb branch and integrability

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Fishnet QFTs: Integrability, periods and beyond

Southapton, 17.07.25

$\mathcal{N}=4$ super-Yang-Mills:

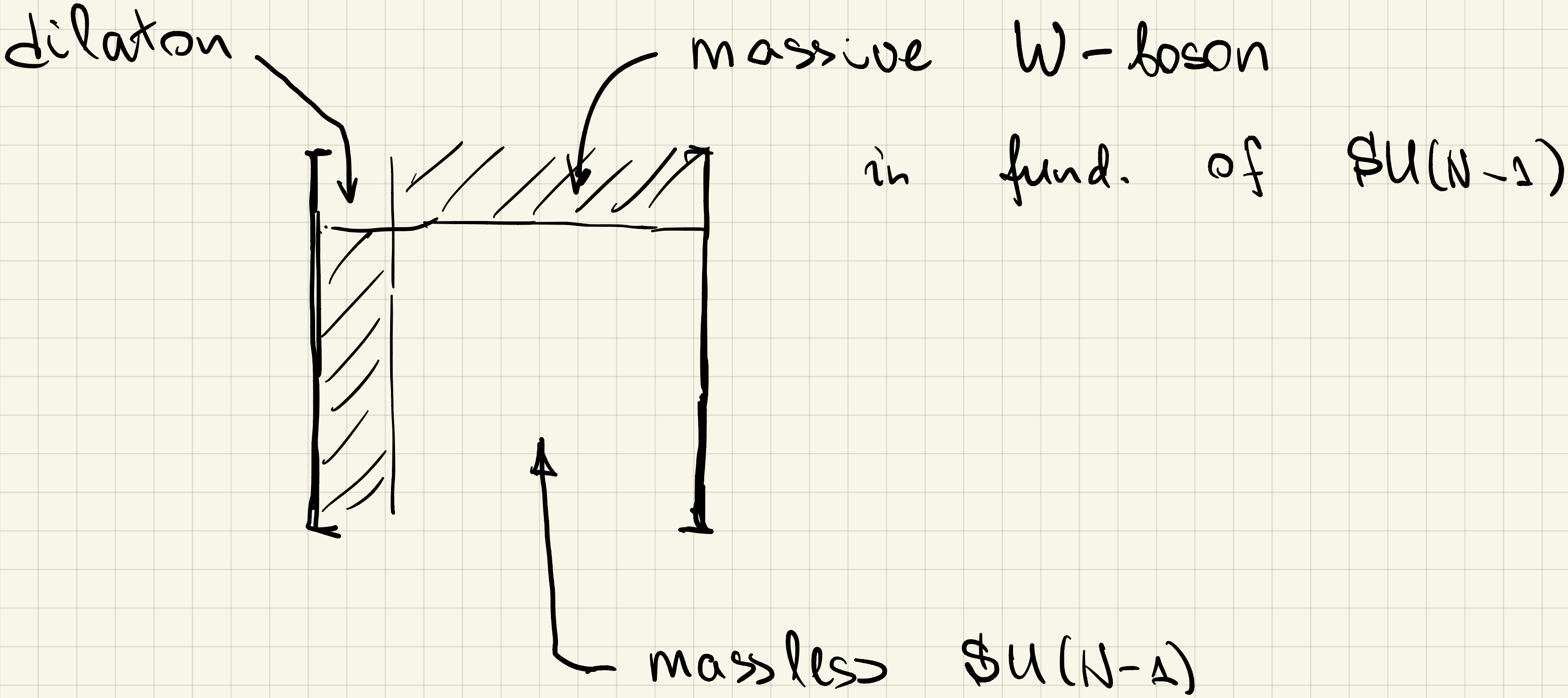
$$\mathcal{L} = \frac{N}{\lambda} \operatorname{tr} \Big\{ -\frac{1}{2} F_{\mu\nu}^2 + \left(D_\mu \Phi_i \right)^2$$

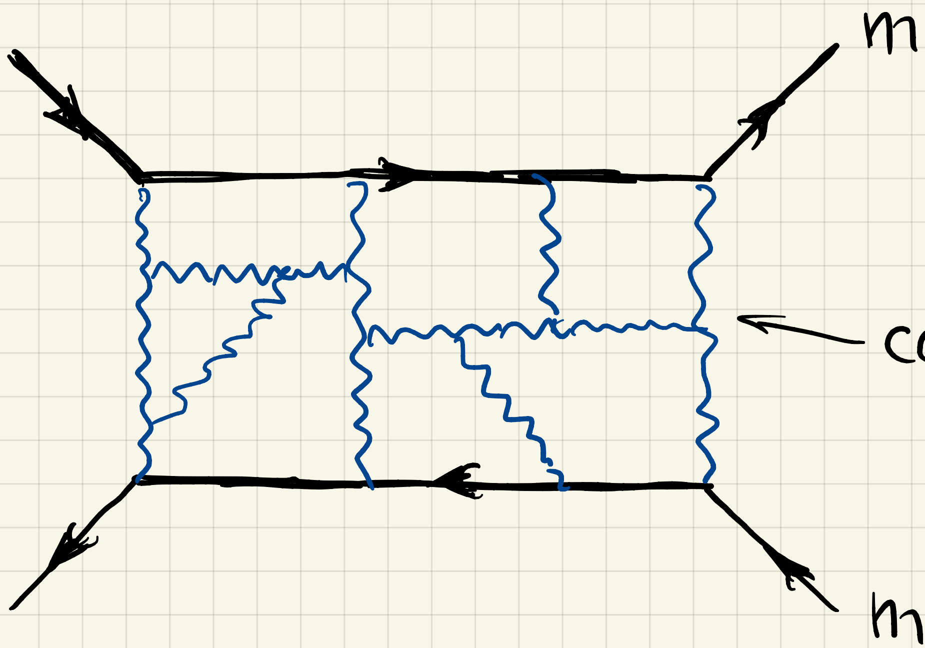
$$+ \frac{1}{2} \left[\Phi_i , \Phi_j \right]^2 + \dots \Big\}$$

Coulomb branch (aka Higgs mechanism):

$$\langle \Phi_i \rangle = \begin{bmatrix} v_i & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$SU(N) \longrightarrow SU(N-1) \times U(1)$$





conformal interactions

- (reasonably) well-defined S-matrix

Alday, Henn, Plefka, Schuster '09

- hydrogen-like bound states

w. known spectrum!

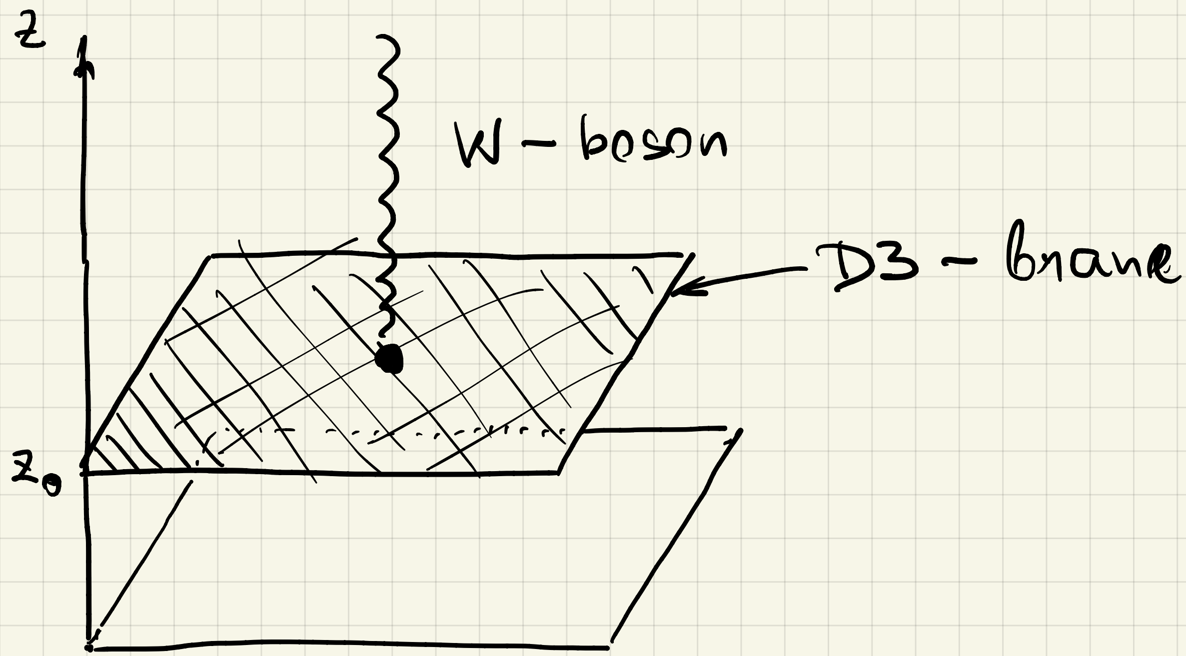
Caron-Huot, Henn '14

- vacuum condensates

← this talk

AdS dual

Skenderis, Taylor '06

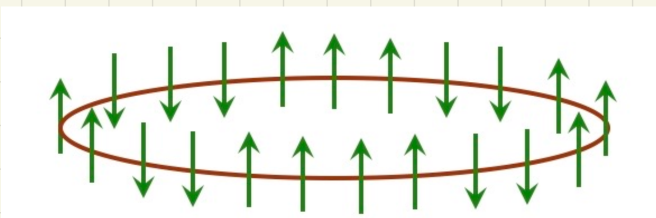


$$v = m_W = \frac{\sqrt{\lambda}}{2\pi} \int_{z_0}^{\infty} \frac{dz}{z^2} = \frac{\sqrt{\lambda}}{2\pi z_0}$$

AdS/CFT as an integrable system

$\lambda \ll 1$

Spin chain

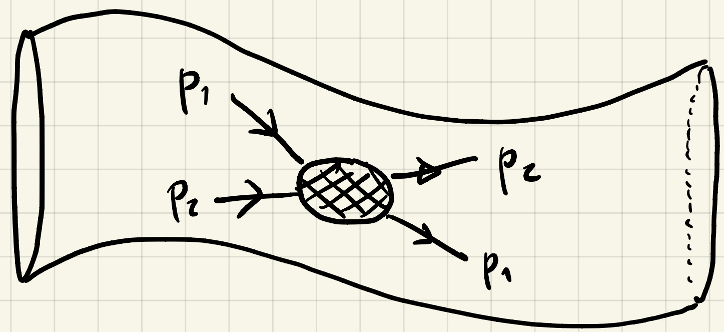


$\mathrm{tr}\, \Phi_{i_1} \ldots \Phi_{i_L}$

$$H = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(2 - 2 P_{l,l+1} + K_{l,l+1} \right)$$

$\lambda \gg 1$

String worldsheet



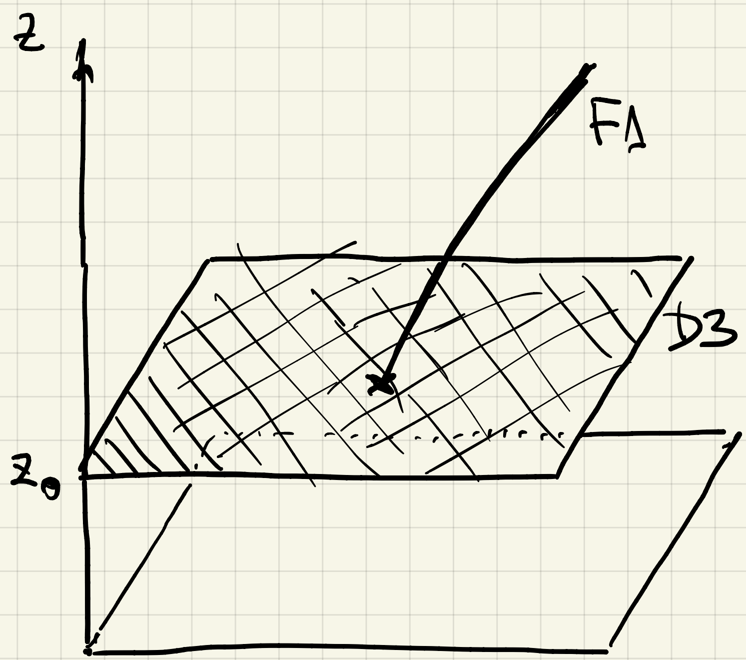
$$S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12}$$

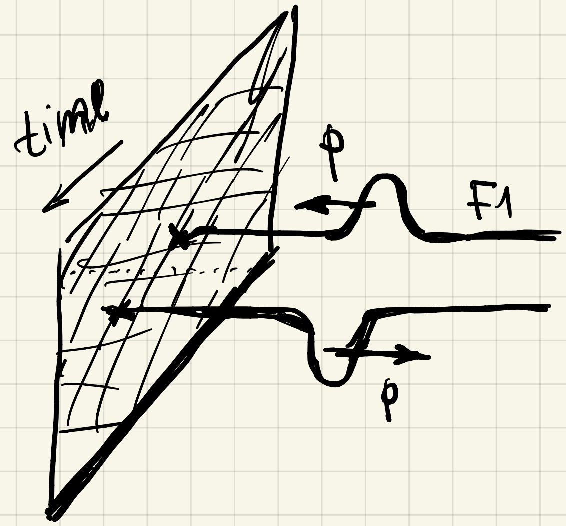
$\boxed{\text{Any } \lambda}$

- Bethe Ansatz
- Quantum spectral curve
- Hexagons

Integrable D-branes

Dekel, Oz '11





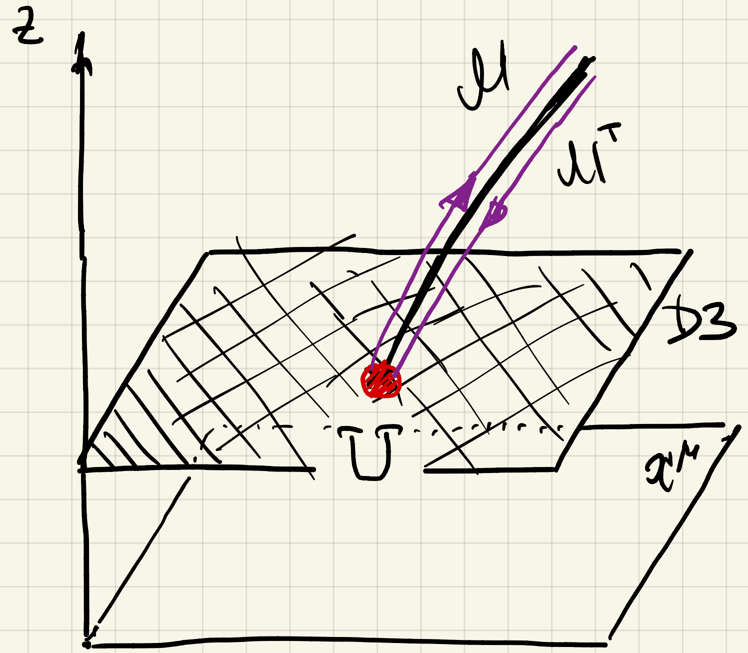
- reflection off the boundary is elastic

$$\Updownarrow$$

Integrability

String integrability on Coulomb branch

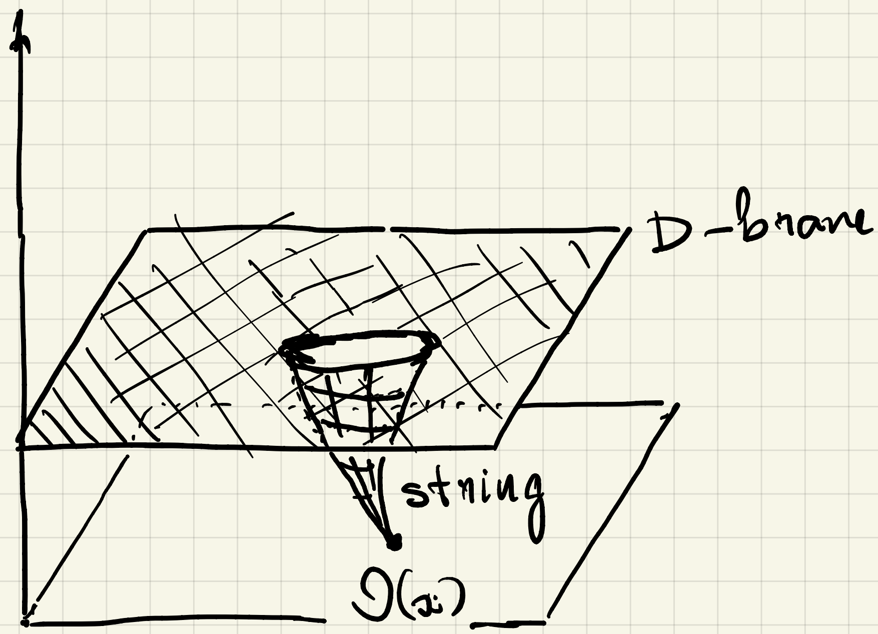
Demjaha, Z. '25



Dynamical reflection matrix:

$$U(\lambda) = \left(X^2 - \lambda^2 Z^2\right)\Pi_- - X^\mu \gamma_\mu + \Pi_+$$

$$\Pi_\pm = \frac{1 \pm \gamma^5}{2}$$

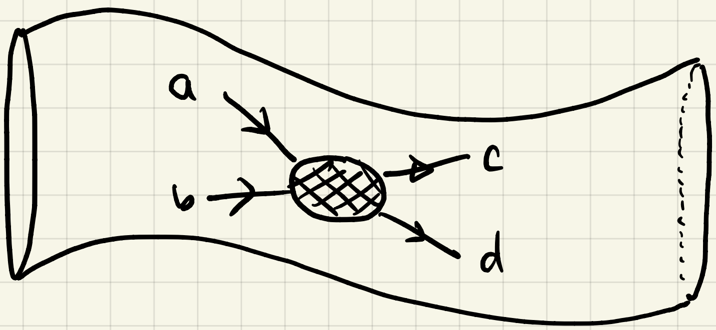


D-brane $\leftrightarrow$ $\langle B_{st}|$ - integrable boundary state

$$\langle \mathcal{O}_n(x) \rangle_{dCFT} = \langle B_{st} | n; x \rangle$$

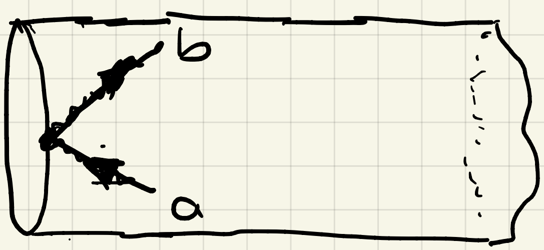
Boundary integrability

"Bulk" S-matrix:



$$S_{ab}^{cd}(p_1, p_2)$$

Boundary reflection amplitude:

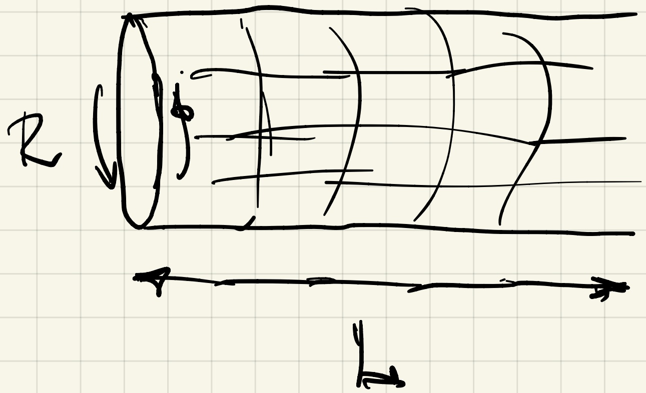


$$K_{ab}(p)$$

Boundary entropy

Affleck, Ludwig '91
Dorey, Fioravanti, Rim, Tateo '04
Pozsgay '10

g - function:



systematically calculable by TBA

$$Z(R,L) \overset{L\to\infty}{\simeq} g(R)\, e^{-E_0 L}$$

condensate $\langle O_n \rangle$

$\Updownarrow$

overlap $\langle B_{st} | n \rangle$

$\Updownarrow$

g-function
in string σ-model

– Condensates play important role in QCD

Shifman, Vainshtein, Zakharov '79

3. Operator expansion and non-perturbative effects

We start the systematic derivation with a discussion of the operator expansion. For the sake of convenience it is divided into two parts: in the present section we consider general problems while computational details are referred to sect. 4. Subsect. 3.1 contains some definitions and generalities which are not specific, in fact, for non-perturbative effects. The principal subsections are 3.2 and 3.3.

3.1. General remarks

We start by introducing notations common to all the cases. Consider the T product of two currents j^A, j^B which can be either light or heavy quark currents. The basic assumption is that at large external momentum q or for a large internal mass m_h the operator expansion [7] is valid:

$$i\int \mathrm{d}x\, e^{iqx}\, \mathrm{T}\{j^A(x), j^B(0)\} = \sum_n C_n^{AB}(q)\, O_n\,, \tag{3.1}$$

where C_n^{AB} are coefficients, O_n are local operators constructed from light quark (u,d,s) or gluon fields. To be more precise, we assume the validity of the expansion only in the few first terms (see subsect. 3.2 for more detail).

The operators O_n are conveniently classified according to their Lorentz spin and dimension d. We will consider only spin-zero operators since only these contribute to the vacuum expectation value. Naturally, the operators in (3.1) satisfy such general requirements as gauge invariance with respect to the gluon field. An important characteristic is operator dimension. An increase in dimension implies extra powers of μ^2/Q^2 or $\mu^2/4m_h^2$ for the corresponding contribution, where μ is some typical hadronic mass entering through the matrix element of O_n. So we list all the operators with zero Lorentz spin and $d \leqslant 6$ *.

$$\begin{aligned}
&I\ (\text{the unit operator})\,, && (d=0)\,,\\
&O_M = \bar\psi M \psi\,, && (d=4)\,,\\
&O_G = G^a_{\mu\nu} G^a_{\mu\nu}\,, && (d=4)\,,\\
&O_\sigma = \bar\psi \sigma_{\mu\nu} t^a \hat M \psi G^a_{\mu\nu}\,, && (d=6)\,,\\
&O_\Gamma = \bar\psi \Gamma_1 \psi \bar\psi \Gamma_2 \psi\,, && (d=6)\,,\\
&O_f = f^{abc} G^a_{\mu\nu} G^b_{\nu\gamma} G^c_{\gamma\mu}\,, && (d=6)\,,
\end{aligned} \tag{3.2}$$

* Other operators can be reduced to those, given in eq. (3.2) plus full derivatives, for example,

$$\bar\psi \gamma_\mu t^a \mathcal{D}_\nu \psi G^a_{\mu\nu} = -\tfrac{1}{2} \bar\psi \gamma_\mu t^a \psi \mathcal{D}_\nu G^a_{\mu\nu} - i\bar\psi \mathcal{D}^2 \gamma_\mu \mathcal{D}_\mu \psi - i\bar\psi \overleftarrow{\mathcal{D}}_\mu \gamma_\mu \mathcal{D}^2 \psi$$

$$+\ \text{full derivatives}\,,$$

and the right-hand side can be expressed in terms of O_M, O_σ by using the equations of motion.

Vacuum condensates

$$\langle \mathcal{O} \rangle = C\, v^{\Delta}$$

Rem Depends on normalization.

$$\mathcal{O} = Z\, \mathcal{O}_b$$

"bare operator"

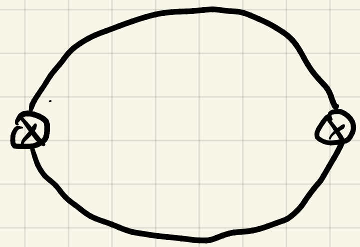
Better definition:

$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O}_b \rangle}{\langle \mathcal{O}_b \mathcal{O}_b \rangle^{1/2}}$$

Example: Konishi operator

$$K = \mathrm{tr}\, \Phi_i \Phi_i$$

$$\boxed{g = \frac{\sqrt{\lambda}}{4\pi}}$$

$$\langle K K \rangle =$$ 

$$D(x) = \frac{\lambda}{8\pi^2 x^2} = \frac{2g^2}{x^2}$$

$$= \underbrace{2}_{\text{symm. factor}} \times \underbrace{6}_{\#\text{ of flavors}} \times (2g^2)^2$$

$$\boxed{\langle K \rangle = \frac{v^2}{4\sqrt{3}\, g^2}}$$

One loop from OPE

Chiral primary: $\quad \mathcal{O} = \mathrm{tr}\, Z^2$

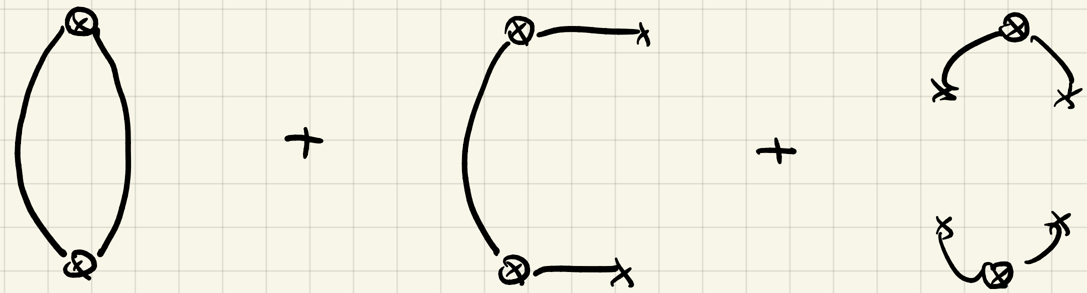
$$Z = \Phi_1 + i\,\Phi_2$$

$$\langle \Phi_1 \rangle = \sigma$$

Baby version of QCD sum rules:

$$\mathcal{O}^\dagger(x)\,\mathcal{O}(0) = \frac{\mathbb{1}}{x^4} + \frac{\mathbb{C}}{x^{4-\Delta_K}}\,K(0) + \dots$$

$$\langle \mathcal{O}^\dagger(x)\, \mathcal{O}(0)\rangle =$$



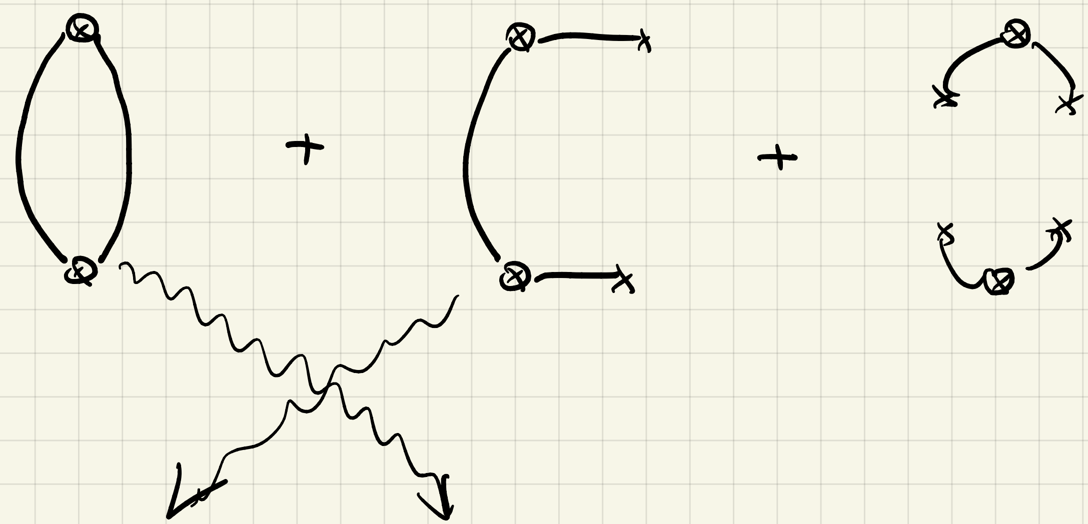
vs.

$$\mathcal{O}^\dagger(x)\,\mathcal{O}(0) = \frac{\mathbb{1}}{x^4} + \frac{C}{x^{4-\Delta_K}}\, K(0) + \dots$$

$$\Delta_K \approx 2$$

$$D_v(x) \overset{x\to 0}{\simeq} \frac{1}{4\pi^2 x^2} + \left(\ln\frac{v x}{2} + \gamma_E - \frac{1}{2} \right) \frac{v^2}{8\pi^2} + \dots$$

$$\langle \mathcal{O}^\dagger(x)\, \mathcal{O}(0) \rangle =$$



$$\frac{2v^2}{x^2}\left(\frac{1}{12 g^2} + \overbrace{\ln\frac{v x}{2} + \gamma_E - \frac{1}{2}} \right)$$

$$\mathcal{O}^\dagger(x)\,\mathcal{O}(0) = \frac{\mathbb{1}}{x^4} + \frac{\mathbb{C}}{x^{4-\Delta_K}}\,K(0) + \dots$$

$$\mathbb{C}_{OOK} = \frac{2}{\sqrt{3}N}\left(1 - 6g^2 + \dots\right)$$

$$\Downarrow$$

$$\frac{1}{x^2}\,\frac{2}{\sqrt{3}}\left[\mathbb{1} - 6g^2 + (\Delta_K - 2)\ln x\right]\langle K\rangle$$

$$2v^2\left(\frac{1}{12g^2} + \ln\frac{vx}{2} + \gamma_E - \frac{1}{2}\right)$$

from tree-level (!) diagrams

$$\parallel$$

$$\frac{2}{\sqrt{3}}\left[1 - 6g^2 + (\Delta_K - 2)\ln x\right]\langle K \rangle$$

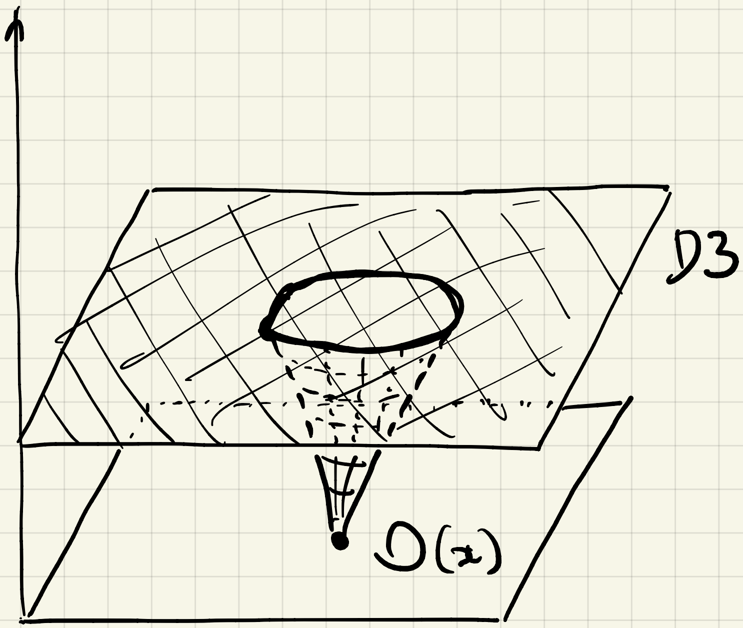
from OPE

Matching coefficients:

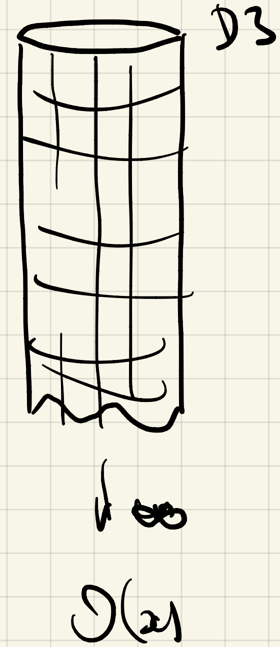
- $\Delta_K = 2 + 12g^2 + \dots$

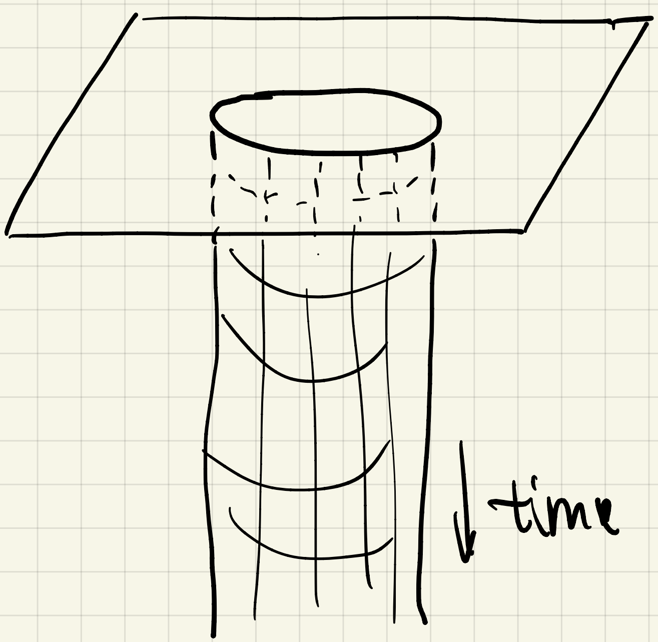
- $\langle K \rangle = \frac{v^2}{4\sqrt{3}g^2}\left[1 + 12g^2\left(\ln\frac{v}{2} + \gamma_E\right) + \dots\right]$

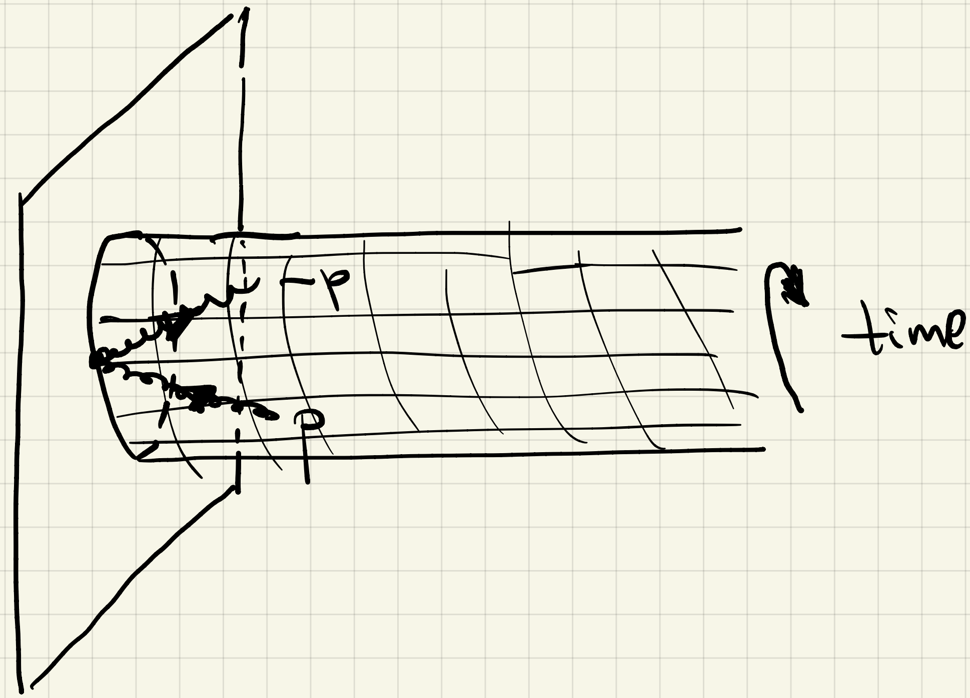
Integrability

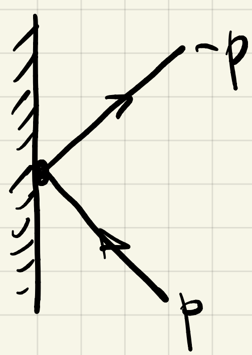


↳ $\langle O \rangle$ = cylinder amplitude









$\sigma_B(p)$ – Boundary reflection phase

Worldsheet kinematics

Magnon dispersion relation:

$$\varepsilon(p) = 1 + \sqrt{1 + 4g\sin^2\frac{p}{2}}$$

Beisert '04

Zhukowski variables:

$$u = g\left(x + \frac{1}{x}\right)$$

$$f^{\pm}(u) \equiv f\left(u \pm \frac{i}{2}\right)$$

$$e^{ip} = \frac{x^+}{x^-}$$

$$\varepsilon = 2 + ig\left(\frac{1}{x^+} - \frac{1}{x^-}\right)$$

Boundary reflection phase

Coronado, Komatsu, Z. '25

$$\sigma_B = \frac{1}{g^2}\left(\frac{u}{2}\right)^{4E(u)} e^{i\chi(x^+) - i\chi(x^-)}$$

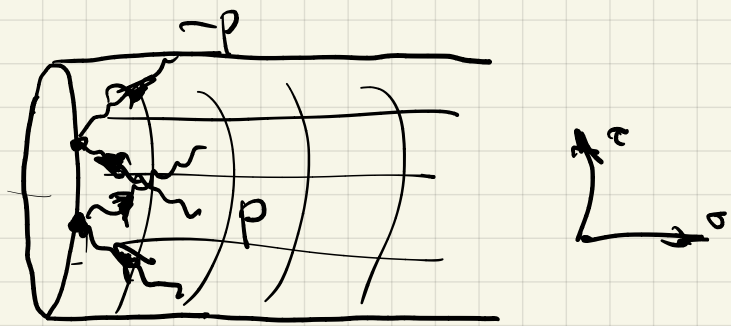
$$\chi(x) = 2i \oint \frac{dz}{2\pi i} \frac{1}{x-z} \oint \frac{dw}{2\pi i w} \ln \frac{\Gamma\left(1 + ig\left(z + \frac{1}{z} + w + \frac{1}{w}\right)\right)}{\Gamma\left(1 - ig\left(z + \frac{1}{z} - w - \frac{1}{w}\right)\right)}$$

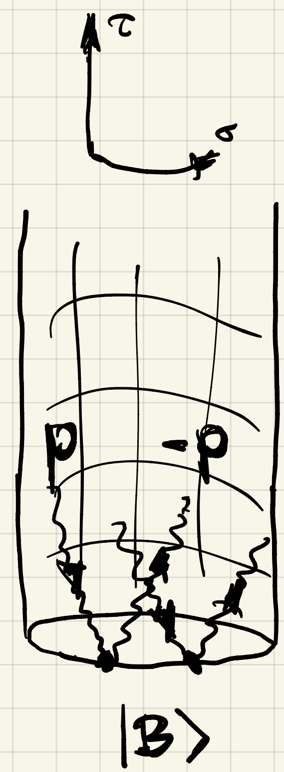
- fixed by analyticity, unitarity & crossing

Integrable boundary states

Ghoshal, Zamolodchikov '93

Piroli, Pozsgay, Vernier '17





Boundary state:

$$\boxed{|B\rangle = \sum C^{B}_{\{p_i,-p_i\}} \,|\{p_i,-p_i\}\rangle}$$

1pt functions as overlaps

D-brane $\leftrightsquigarrow$ $\langle Bst|$ - boundary state

$$\langle O_n(x)\rangle_{dCFT} = \langle Bst|n;x\rangle$$

$$\langle\{p_j,-p_j\}|$$

Overlap formula:

$$\langle O_n(x)\rangle_{dCFT} = \sqrt{\prod_j f(p_j)\ \mathrm{Sdet}\, G}$$

G - Gaudin matrix

universal
doesn't depend on $\langle BST|$

Weak coupling:

$$\sigma_B(u) \simeq \frac{u^2}{4g^2}\left(1 + 4g^2\,\frac{\gamma_E + \ln\frac{u}{2}}{u^2+\frac{1}{4}} + \dots\right)$$

$$\langle O \rangle \simeq \left(\frac{u}{2g}\right)^{L-M} \sqrt{\prod_{j=1}^{M} \sigma_B(u_j)\ \ \mathrm{Sdet}\, G}$$

Example: Konishi operator

$$\{u_j\} = \left\{\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}}\right\} \qquad \mathrm{Sdet}\, G = \frac{1}{3}$$

$$\sigma_B(u) \simeq \frac{v^2}{4g^2}\left(1 + 4g^2\,\frac{\gamma_E + \ln\frac{v}{2}}{u^2+\frac{1}{4}} + \dots\right)$$

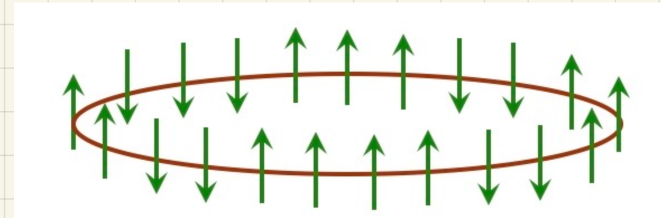
$$\overset{u=\frac{1}{\sqrt{12}}}{=} \frac{v^2}{4g^2}\left[1 + 12g^2\left(\gamma_E + \ln\frac{v}{2}\right)\right]$$

$$\langle K \rangle = \frac{v^2}{4\sqrt{3}\,g^2}\left[1 + 12g^2\left(\gamma_E + \ln\frac{v}{2}\right)\right]$$

- agrees w. direct calculation.

Boundary state

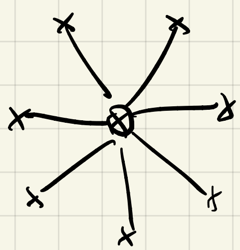
$$\mathcal{O} = \Psi^{i_1 \dots i_L} \,\mathrm{tr}\, \Phi_{i_1} \dots \Phi_{i_L}$$



$$\Phi_i \to \langle \Phi_i \rangle = v_i$$

- projects all spins onto a given axis

$\langle \mathcal{O} \rangle =$



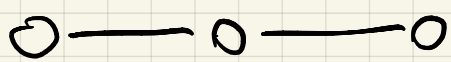
$$= \left(\frac{v^2}{2g^2}\right)^{\frac{L}{2}} \frac{1}{\sqrt{L}} \frac{\langle \mathrm{Bst} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{1/2}}$$

$$\boxed{\mathrm{Bst}_{i_1, \dots, i_L} = n_{i_1} \dots n_{i_L}} \qquad n_i = \frac{v_i}{|v|}$$

Bethe Ansatz

$$\left(\frac{u_{aj} - \frac{i q_a}{2}}{u_{aj} + \frac{i q_a}{2}}\right)^L \prod_{bk} \frac{u_{aj} - u_{bk} + \frac{i M_{ab}}{2}}{u_{aj} - u_{bk} - \frac{i M_{ab}}{2}} = -1$$

$$M = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



$$\gamma = \frac{\lambda}{8\pi^2} \sum_{aj} \frac{q_a^2}{u_{aj}^2 + \frac{q_a^2}{4}}$$ - Anomalous dimension

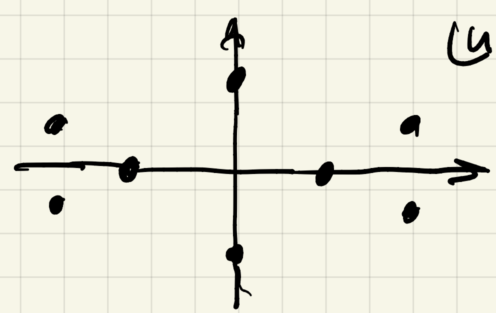
$\langle Bst|$ is integrable:

$$\langle Bst | \{u_{aj}\} \rangle = 0$$

unless

$$\{u_{aj}\} = \{u_{aj}, -u_{aj}\}\Big|_{j=1}^{\frac{K_a}{2}}$$

$\mathbb{Z}_2$ symmetry: $\{u_{aj}\} = \{-u_{aj}\}$



Overlap formula

$$\langle \mathcal{D} \rangle_\tau = \left(\frac{v^2}{2g^2}\right)^{\frac{L}{2}} \sqrt{\frac{\prod_j u_{2j}^2 \left(u_{2j}^2 + \frac{1}{4}\right)}{L \prod_j u_{1j}^2 \left(u_{1j}^2 + \frac{1}{4}\right) \prod_j u_{3j}^2 \left(u_{3j}^2 + \frac{1}{4}\right)} \frac{\det G^+}{\det G^-}}$$

de Leeuw, Gombor, Kristjansen, Linardopoulos, Pozsgay '19

Gaudin superdeterminant:

$$G^\pm_{aj,bk} = \left(\frac{L\, q_a}{u_{aj}^2 + \frac{q_a^2}{4}} - \sum_{cl} K^+_{aj,cl} \right) \delta_{ab} \delta_{jk} + K^\pm_{aj,bk}$$

$$K^\pm_{aj,bk} = \frac{M_{ab}}{(u_{aj} - u_{bk})^2 + \frac{M_{ab}^2}{4}} \pm \frac{M_{ab}}{(u_{aj} + u_{bk})^2 + \frac{M_{ab}^2}{4}}$$

Gaudin superdeterminant

Bethe eqs: $e^{i\phi_j} = 1$ / e.g. $\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L \prod_{k\neq j} \frac{u_j - u_k - i}{u_j - u_k + i} \approx 1$ /

Gaudin matrix: $G_{jk} = \frac{\partial \phi_j}{\partial u_k}$

Boundary integrability requires $u_{\Omega(j)} = -u_j$

Superdeterminant: $\mathrm{Sdet}\, G = e^{\mathrm{tr}\, \Omega \ln G}$

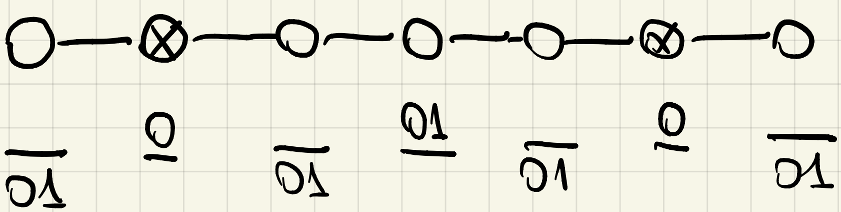
$$G = \begin{array}{c c} & \begin{array}{cc} \{u\} & \{-u\} \end{array} \\ \begin{array}{c} \{u\} \\ \{-u\} \end{array} & \begin{bmatrix} A & B \\ B & A \end{bmatrix} \end{array}$$

$$\det G = \det(A+B)\det(A-B)$$

$$\boxed{\mathrm{Sdet}\, G = \frac{\det(A+B)}{\det(A-B)}}$$

Generic operators and higher loops

$$\langle \mathcal{O}(\alpha)\rangle \sim \sqrt{\frac{\prod_{a_i} Q_{a_i}(i\alpha_{a_i}/2)}{\prod_{b_j} Q_{b_j}(i\alpha_{b_j}/2)}\ \frac{\det G^+}{\det G^-}}$$



The all-loop formula:

- add reflection phase for each magnon
- replace 1-loop BAE by non-perturbative ABA

Conclusions

- Coulomb branch is integrable

  Demjaha, Z '25

- Vacuum condensates are calculable non-perturbatively

  Coronado, Komatsu, Z '25

- So far all results are asymptotic (valid up to wrapping corrections)

  TBA? QSC?

- Other observables: spectrum? S-matrix?